Problems

0 If you are thinking of/planning to do a project, I need a brief but clear and specific proposal from you via email as soon as possible. Tell me what you are thinking and I’ll get back to you with questions and suggestions. Even if you told me something about what you want to do earlier this term, tell me more specifically now!

1. Do Exercise 85, p. 96. Assume that the whole universe consists of dragons. Use the predicates: $H(x)$, $G(y)$, $F(z)$, $C(v,w)$ to mean $x$ is happy, $y$ is green, $z$ can fly, and $v$ is a child of $w$, respectively.

2. Do Exercise 86, p. 96.

3. Prove using the FO Compactness Theorem that for any first-order vocabulary, $Σ$, there is no formula, $ϕ_f$ that says, “My universe is finite”. [Hint: assume for the sake of a contradiction that $ϕ_f$ means, “My universe if finite”. In particular, for each $n \in \mathbb{N}^+$, there is a structure $A \models ϕ_f$ with $|A| = n$. Furthermore for every $A \models ϕ_f$, $|A|$ is finite. Use the compactness theorem to derive a contradiction.]

4. Let $ϕ$ be a first order sentence of vocabulary $Σ = \langle P^{a_1}, \ldots, P^{a_k}; f^{r_1}, \ldots, f^{r_t} \rangle$. Suppose that $ϕ$ is complete, i.e., for all sentences, $α \in L(Σ)$, either $ϕ \vdash α$ or $ϕ \vdash ¬α$. Argue that the theory($ϕ$) is a decidable set, where,

$$\text{theory}(ϕ) = \{α \text{ a sentence in } L(Σ) \mid ϕ \vdash α\} .$$

5. Note that by the Fundamental Theorem of Ehrenfeucht-Fraïssé games, $A \equiv B$ iff for all $m$, $A \sim_m B$. Define $A \sim_∞ B$ to mean that Delilah has a winning strategy for the infinite game on $A$ and $B$, i.e., the game goes on forever and Delilah must never lose after any step.

(a) Show that if $A$ and $B$ have countable universes and $A \sim_∞ B$, then $A \cong B$.

(b) Define the first order sentence DLOWE to be the conjunction of the following axioms. (Recall that $x < y$ is an abbreviation for $x ≤ y ∧ x ≠ y$.)

Dense: $∀xy∃z (x < y → x < z ∧ z < y)$

Reflexive: $∀xx ≤ x$

AntiSymmetric: $∀xy(x ≤ y ∧ y ≤ x → x = y)$

Transitive: $∀xyz(x ≤ y ∧ y ≤ z → x ≤ z)$

Total: $∀xy(x ≤ y ∨ y ≤ x)$

Without Endpoints: $∀x ∃yz (y < x ∧ x < z)$

Show that for any two countable models, $A, B$ of DLOWE, $A \sim_∞ B$. You should devise a winning strategy for Delilah and explain why it is a winning strategy.

Thus DLOWE is $\aleph_0$ categorical, i.e., it has a unique countable model up to isomorphism.

(c) Show that any $\aleph_0$-categorical sentence is complete.