Draft of Final Crib Sheet

Def of Prop Logic Formulas ($\mathbf{P_{fmla}}$) $\mathbf{P_{var}} \stackrel{\text{def}}{=} \{p, q, r, s, p_0, q_0, r_0, s_0, p_1, q_1, r_1, s_1, \ldots\}$

base: $\top, \bot \in \mathbf{P_{fmla}}$; if $a \in \mathbf{P_{var}}$ then $a \in \mathbf{P_{fmla}}$; **ind:** If $\alpha, \beta \in \mathbf{P_{fmla}}$ then $\neg\alpha, (\alpha \vee \beta) \in \mathbf{P_{fmla}}$

Completeness Thm for Prop Logic: If φ is unsatifiable then $\varphi \vdash \Box$.

FO Syntax: term(Σ) : VAR $\stackrel{\text{def}}{=} \{x, y, z, u, v, w, x_0, y_0, \ldots, x_1, y_1, \ldots\}$

base: If $v \in$ VAR then $v \in$ term(Σ); **ind:** If $t_1, t_2, \ldots, t_r \in$ term(Σ); $f^r \in \Sigma$ then $f(t_1, \ldots, t_r) \in$ term(Σ)

FO formulas: $\mathcal{L}(\Sigma)$ **atomic fmla:** If $t_1, \ldots, t_a \in$ term(Σ) and $P^a \in \Sigma$ then $P(t_1, \ldots, t_a), t_1 = t_2 \in \mathcal{L}(\Sigma)$.

ind: If $\alpha, \beta \in \mathcal{L}(\Sigma)$ and $v \in$ VAR then $\neg\alpha,\ (\alpha \vee \beta),\ \exists v(\alpha) \in \mathcal{L}(\Sigma)$

Tarski's Definition of Truth: $\mathcal{A} \in$ STRUC[Σ]

terms: base: If $v \in$ VAR then $v^{\mathcal{A}} \in |\mathcal{A}|$ **ind:** $t_1, \ldots, t_r \in$ term(Σ); $f^r \in \Sigma,\ f(t_1, \ldots, t_r)^{\mathcal{A}} \stackrel{\text{def}}{=} f^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_r^{\mathcal{A}})$

atomic fmla: $\mathcal{A} \models P(t_1, \ldots, t_a)$ iff $(t_1^{\mathcal{A}}, \ldots, t_{a_i}^{\mathcal{A}}) \in P^{\mathcal{A}}$; $\mathcal{A} \models t_1 = t_2$ iff $t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$

ind: $\mathcal{A} \models \neg\alpha$ iff $\mathcal{A} \not\models \alpha$; $\mathcal{A} \models (\alpha \vee \beta)$ iff $\mathcal{A} \models \alpha$ or $\mathcal{A} \models \beta$; $\mathcal{A} \models \exists v(\alpha)$ iff exists $a \in |\mathcal{A}|$, $\mathcal{A}[a/v] \models \alpha$

Convert FO Fmla to Equivalent Fmla in Rectified Prenex Normal (RPF) Form; then Skolemize

1. Remove all "$\rightarrow$"s using the fact that $\alpha \rightarrow \beta \equiv \neg\alpha \vee \beta$.
2. Push all "$\neg$"s all the way inside using de Morgan and quantifier rules:

   $\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$; $\neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta$; $\neg\forall x(\varphi) \equiv \exists x(\neg\varphi)$; $\neg\exists x(\varphi) \equiv \forall x(\neg\varphi)$
3. **Rectify** by renaming bound variables so each occurs only once and no bound variable also occurs free.
4. Pull quantifiers out using the following rules, assuming that x does not occur in α:

   $\alpha \wedge \forall x(\beta) \equiv \forall x(\alpha \wedge \beta)$; $\alpha \vee \forall x(\beta) \equiv \forall x(\alpha \vee \beta)$; $\alpha \wedge \exists x(\beta) \equiv \exists x(\alpha \wedge \beta)$; $\alpha \vee \exists x(\beta) \equiv \exists x(\alpha \vee \beta)$
5. **Skolemize:** remove existential quantifier $\exists x$ and replace x by $f(u_1, \ldots, u_r)$ where $\forall u_1 u_r$ are to the left of $\exists x$ and f a new function symbol.

Semantic Implication: For $\alpha, \beta \in \mathcal{L}(\Sigma)$, $\alpha \models \beta \quad \Leftrightarrow \quad \forall \mathcal{A} \in \text{STRUC}[\Sigma](\mathcal{A} \models \alpha \ \rightarrow\ \mathcal{A} \models \beta)$.

FO Resolution Proof: For $\alpha, \beta \in \mathcal{L}(\Sigma)$, $\alpha \vdash \beta \quad \Leftrightarrow \quad \varphi_S \vdash \Box$ i.e., to prove β from α we let $\varphi := \alpha \wedge \neg\beta$, put φ into RPF, skolemize, and derive the empty clause from φ_S using unification and resolution.

Gödel's Completeness Thm for FO Logic: For any $\alpha, \beta \in \mathcal{L}(\Sigma)$, $\alpha \models \beta \quad \Leftrightarrow \quad \alpha \vdash \beta$.

Def. $S \subseteq \{0,1\}^*$ is Recursive if χ_S is computable and r.e. if p_S is computable. **Thm** Recursive $=$ r.e. $\cap$ co-r.e..

$$\chi_S(w) \quad = \quad \begin{cases} 1 & \text{if} \quad w \in S \\ 0 & \text{if} \quad w \notin S \end{cases} \qquad\qquad p_S(w) \quad = \quad \begin{cases} 1 & \text{if} \quad w \in S \\ \nearrow & \text{if} \quad w \notin S \end{cases}$$

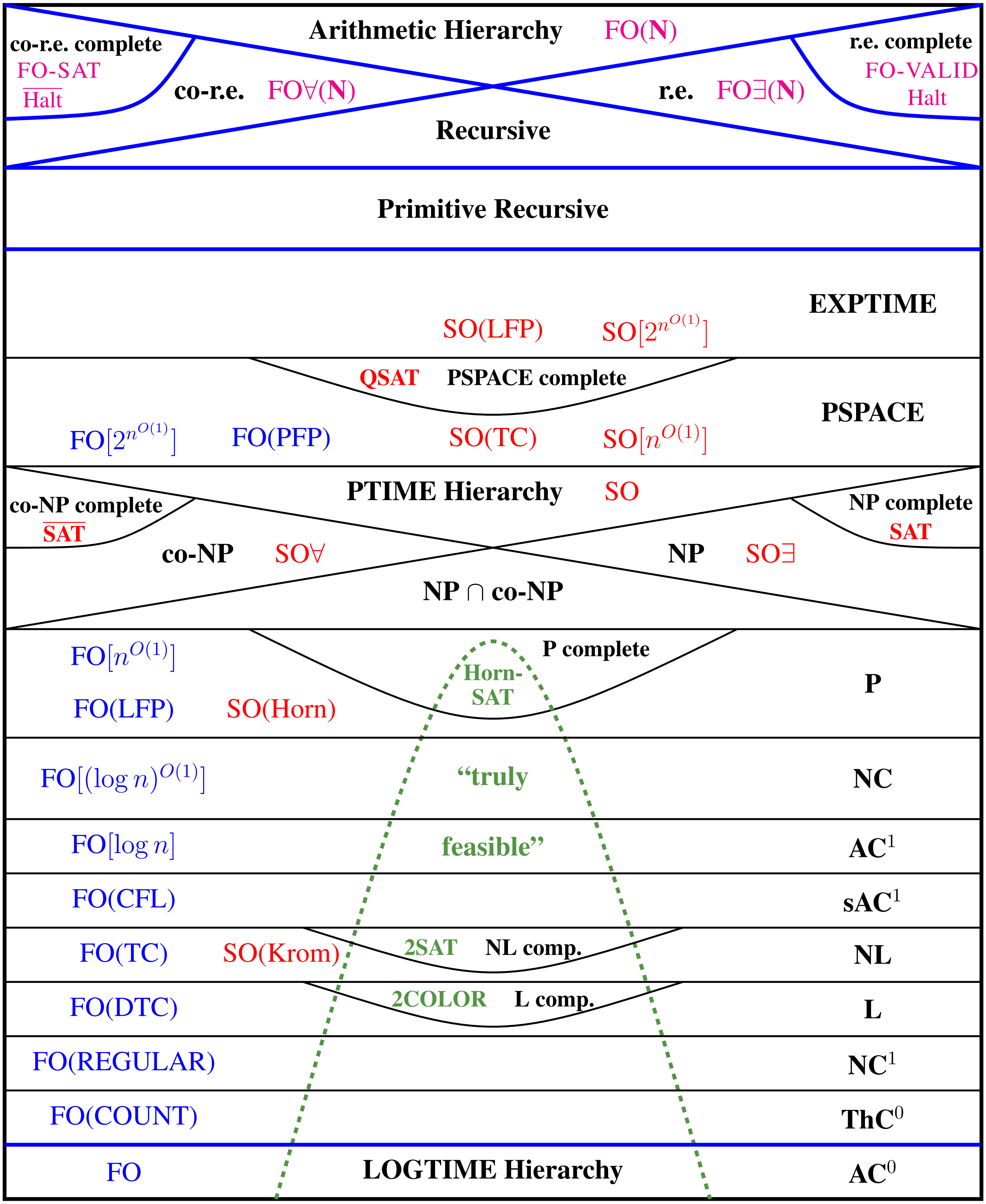
Gödel's Incompleteness Thm for FO Logic: Theory(**N**) is not r.e., i.e., not axiomatizable.

$\mathcal{A}, \mathcal{B} \in$ STRUC[Σ] are **isomorphic** ($\mathcal{A} \cong \mathcal{B}$) iff exists $\eta : |\mathcal{A}| \xrightarrow[\text{onto}]{1:1} |\mathcal{B}|$ such that

$$\text{forall } P^a \in \Sigma, e_1, \ldots, e_a \in |\mathcal{A}| \quad \big((e_1, \ldots, e_a) \in P^{\mathcal{A}} \ \Leftrightarrow \ (\eta(e_1), \ldots, \eta(e_a)) \in P^{\mathcal{B}} \big) \qquad \wedge$$
$$\text{forall } f^r \in \Sigma, e_1, \ldots, e_r \in |\mathcal{A}| \quad \big(\eta(f^{\mathcal{A}}(e_1, \ldots, e_r)) \ = \ f^{\mathcal{B}}(\eta(e_1), \ldots, \eta(e_r)) \big)$$

In the EF Game, $\mathcal{G}_m^k(\mathcal{A}, \mathcal{B})$, **Delilah wins** ($\mathcal{A} \sim_m^k \mathcal{B}$) if after every step, $j \leq m$ the function, η_j that maps $c^{\mathcal{A}} \mapsto c^{\mathcal{B}}$ for constant symbols $c \in \Sigma$ and $x_i^{\mathcal{A}} \mapsto x_i^{\mathcal{B}}$ for all pebbles x_i that have been placed so far, is an isomorphism of the induced substructures. **Samson wins** if at any step, η_j is not an isomorphism.

Fund Thm of EF Games: Finite relational, $\Sigma, \mathcal{A}, \mathcal{B} \in$ STRUC[Σ], $\mathcal{A} \equiv_m^k \mathcal{B} \Leftrightarrow \mathcal{A} \sim_m^k \mathcal{B}$


Arithmetic Hierarchy
FO(N)
co-r.e. complete
FO-SAT
Halt
co-r.e.
FO∀(N)
r.e.
FO∃(N)
r.e. complete
FO-VALID
Halt
Recursive
Primitive Recursive
EXPTIME
SO(LFP)
SO[$2^{n^{O(1)}}$]
QSAT
PSPACE complete
PSPACE
FO[$2^{n^{O(1)}}$]
FO(PFP)
SO(TC)
SO[$n^{O(1)}$]
PTIME Hierarchy
SO
co-NP complete
SAT
co-NP
SO∀
NP
SO∃
NP complete
SAT
NP ∩ co-NP
FO[$n^{O(1)}$]
P complete
Horn-SAT
P
FO(LFP)
SO(Horn)
FO[$(\log n)^{O(1)}$]
"truly
NC
FO[$\log n$]
feasible"
AC^1
FO(CFL)
sAC^1
FO(TC)
SO(Krom)
2SAT
NL comp.
NL
FO(DTC)
2COLOR
L comp.
L
FO(REGULAR)
NC^1
FO(COUNT)
ThC^0
FO
LOGTIME Hierarchy
AC^0