11.1 Compactness Theorem for First Order Logic

Last time we proved Gödel's Completeness Theorem for First Order Logic. The following is a corollary.

Theorem 11.1 (Compactness of FO Logic) *Let $\Gamma \subseteq \mathcal{L}(\Sigma)$ be a collection of first order formulas. If Γ is finitely satisfiable (that is, every finite subset of Γ is satisfiable), then Γ is satisfiable.*

Proof: We show the contrapositive. Assume that Γ is not satisfiable. Since Γ has no models, it follows vacuously that $\Gamma \models \bot$. Thus, by the Completeness Theorem, $\Gamma \vdash \bot$. Consider the proof by resolution of $\Gamma \vdash \bot$. It takes finitely many clauses from Γ and repeatedly applies substitution and resolution, deriving $\Box$. Since the process is finite, only some finite subset $\Gamma_0 \subseteq \Gamma$ is used in the proof. Thus $\Gamma_0 \vdash \bot$. By Soundness, this means that $\Gamma_0 \models \bot$. Thus, Γ_0 is not satisfiable, i.e., Γ is not finitely satisfiable, $\Box$

11.2 Applications of First-Order Compactness

Let $\Sigma_{\text{graph}} = (E^2;)$ be the standard graph vocabulary.

Proposition 11.2 *There is no first order formula φ that expresses the property of graph-connectedness. More formally, there does not exist φ such that Mod$(\varphi) = \{G \in \text{STRUC}[\Sigma_{graph}] \mid G \text{ is connected}\}$, where Mod$(\varphi)$ is the set of all models satisfying φ.*

Proof: Suppose φ_c is a formula expressing graph-connectedness as described above. Let $\Gamma = \{\varphi_c, \text{dist}_{\geq 1}(s,t), \text{dist}_{\geq 2}(s,t), \ldots\}$. Γ is unsatisfiable, because in order to satisfy it, a graph must have two vertices s and t that have $\text{dist}(s,t) \geq n$ for every natural n, while also being connected. But every finite subset of Γ is satisfiable; for example, let $\Gamma_0 = \{\text{dist}_{\geq n_1}(s,t), \text{dist}_{\geq n_2}(s,t), \ldots, \text{dist}_{\geq n_k}(s,t)\}$. Let $N = max(n_1, n_2, \ldots, n_k)$, and then any model satisfying $\{\text{dist}_{\geq N}(s,t), \varphi_c\}$ satisfies Γ_0.

Note that for each $k \in \mathbf{Z}^+$, we can write $\text{dist}_{\geq k}(x,y) \stackrel{\text{def}}{=} \neg p_k(x,y)$ in $\mathcal{L}(\Sigma_{\text{graph}}$, where $p_k(x,y)$ says that there is a path of length at most k from x to y:

$$\begin{aligned} p_1(x,y) &\stackrel{\text{def}}{=} x = y \vee E(x,y) \\ p_{k+1}(x,y) &\stackrel{\text{def}}{=} \exists z(p_1(x,z) \wedge p_k(z,y)) \end{aligned}$$

By compactness, since every finite subset of Γ is satisfiable, Γ should be satisfiable, so the fact that it is not contradicts our assumption that such a φ_c exists. $\Box$

As another application, we construct a "non-standard model of arithmetic." We denote by **N** the standard model of arithmetic, with universe $\{0, 1, 2, \ldots\}$, and $\Sigma_{\text{\#-thy}} = \{\leq; 0, 1, +, *\}$ where all functions and predicates are defined in the expected way. We denote the set of formulas satisfied by this model $Th(\mathbf{N})$ (read "the theory of **N**"), $Th(\mathbf{N}) = \{\varphi \in \mathcal{L}(\Sigma_{\text{\#-thy}}) \mid \mathbf{N} \models \varphi\}$.

Now define $\Gamma = Th(\mathbf{N}) \cup \{0 \leq c, 1 \leq c, 2 \leq c, \ldots\}$. Here we are using abbreviations: $2 \hookrightarrow 1 + 1$, $3 \hookrightarrow 1 + 2, \ldots$.

Γ is finitely satisfiable, because for any finite subset of formulas we can choose **N** as our model, which will contain some c large enough to satisfy all of the finitely many added formulas. Then by compactness, Γ is satisfiable, so has some model $\mathbf{N}^*$, which we call a "non-standard model of $Th(\mathbf{N})$."

We can think of this model as **N**, along with a new element c larger than any elements of **N**. We must also include elements like $c-1, c+1, 2*c, c^2, \lfloor c/2 \rfloor, \lfloor \sqrt{c} \rfloor$, etc. As $Th(\mathbf{N}) \subseteq \Gamma$, and $\mathbf{N}^* \models \Gamma$, it follows that $\mathbf{N}^* \models Th(\mathbf{N})$, so all "normal" facts about the natural numbers are also satisfied by this model.

This also tells us that any attempt to define a specific model expressing what we think of as the natural numbers is not possible using only first-order axioms, as any such set of axioms is also satisfied by some non-standard model.

One other suprising consequence of competeness is the following

Theorem 11.3 (Löwenheim-Skolem Theorem) *For any (countable) vocabulary, Σ, and any $\Gamma \subseteq \mathcal{L}(\Sigma)$, if Γ is satisfiable, then Γ has a countable model.*

Proof: As we have seen we can translate Γ to an equi-satisfiable set of universal sentences. It then follows from Herbrand's Theorem that if Γ is satisfiable, then Γ has a Herbrand model, $\mathcal{H}$. But, $|\mathcal{H}|$ is the set of closed terms of Σ, which is a countable set. $\square$

Recall that it is well believed that ZFC formalizes all of standard mathematics. We thus get the surprising corollary:

Corollary 11.4 *If ZFC is satisfiable, then ZFC has a countable model, $\mathcal{M}$.*

This is surprising because such a countable model, $\mathcal{M}$, must contain sets such as the real numbers, **R**, the power set of the reals, $\wp(\mathbf{R})$, etc. These must "really" be countable, even though in $\mathcal{M}$ they seem uncountable.