Cooperation: Students should talk to each other about the subject matter of this class and help each other. It is fine to discuss the problems and ask questions about them. I encourage such questions in class, office hours, and on Piazza. However, there is a line past which you must not go, e.g., sharing or copying a solution is not okay and could result in failure. If a significant part of one of your solutions is due to someone else, or something you've read then **you must acknowledge your source!** Failure to do so is a serious academic violation, likely to result in failure of the course or worse. Furthermore, **all solutions must be written by yourself, in your own words**. You may get an idea from somewhere or someone and acknowledge that, but you must still understand it and explain it yourself. A copied solution, even with the source acknowledged will be considered plagiarism. The exception is if it is in quotation marks and cited specifically. But in this case, don't bother because you won't get credit for quoting someone else's solution.

(When commenting about hw problems on Pizza before solutions have been handed back, please do not give too much away. Only ask or comment about the meaning of the question and its relation to readings, handouts or lectures. Do not suggest an approach in any detail. If you are not sure whether your question or answer might give too much away, please send it privately to the instructors, and we will let you know. Thanks.)

General Strategy for doing the homeworks: My desire for the homeworks is that they help you **understand** the concepts from lecture and readings, i.e., that these concepts not only seem believable, but you can employ them. Please read the homework early and make sure that you understand all the questions.

For almost all the problems I give, you should be able to look at a few tiny examples, and try to solve the problem on those examples. If you can do it for the tiny examples, then you are part way to giving a rule that will solve the problem in general. If you cannot solve or are confused by what the problem means on one of your small examples, then that would be a great time to ask me or post a question about it.

Reading Please read the whole first chapter of Schöning by the time this assignment is due. I suggest that you only read one section at a sitting and that you think about all of the exercises – assigned or not – to help make sure that you are understanding all the concepts as you read.

Note about this problem set: It is most important to me that you understand the text and work through the examples, e.g., you can use truth tables to solve the logical puzzles and you can convert a formula to an equivalent one in CNF. Once you can do that, then work on these problems. Don't worry if you can't solve all of them, but start early and ask questions if you feel you are not making progress.

Class notes or text? When I don't exactly follow the text I would prefer if you use the definitions I make in class rather than those in the text, but please point out to me if there is an interesting issue, or if something would be easier if you follow the way the book did it. In particular, read the handouts I post on the syllabus page and consider them authoritative. For example, please use my definitions of $\mathbf{P_{var}}$ and $\mathbf{P_{fmla}}$ rather than Schöning's.

Extra Credit for Helpful Posts: A good, helpful answer on Piazza to a question posted by someone else, or pointing out a typo in any of my handouts will be very much appreciated and also get you some extra credit. This is a way you can help everyone succeed.

Problems

1. For each of the following, state whether it is true or false. If false, give a counterexample. If true, give a convincing argument why it is true. For all $\alpha, \beta \in \mathbf{P_{fmla}}$,
   (a) if $(\alpha \rightarrow \beta)$ is valid and α is valid, then β is valid.
   (b) if $(\alpha \rightarrow \beta)$ is satisfiable and α is satisfiable, then β is satisfiable.
   (c) if $(\alpha \rightarrow \beta)$ is valid and α is satisfiable, then β is satisfiable.

2. Prove the following Proposition: If $\mathcal{A}$ and $\mathcal{B}$ are suitable truth assignments for propositional formula α, and $\mathcal{A}$ and $\mathcal{B}$ agree on all the elements of $\mathbf{P_{var}}$ that occur in α, then $\mathcal{A}(\alpha) = \mathcal{B}(\alpha)$.

   [Hint: please prove this by induction on the structure of the formula α. This means, you should first prove it for the two base cases. Next, assuming the proposition holds for α_1 and α_2, you should prove that it also holds for $\neg\alpha_1$ and for $(\alpha_1 \vee \alpha_2)$.]

3. Horn formulas: Read the Horn-SAT algorithm from the text and apply it to a few examples, so that you understand how it works.
   (a) In particular, apply the algorithm to the following two Horn formulas and state whether they are satisfiable or not. Don't hand in your work, just provide the satisfying assignment you derived from the algorithm if it is SAT and say "UNSAT" otherwise.[1]
      i. $\alpha_1 \stackrel{\text{def}}{=} \{C_1, C_2, C_3, C_4, C_5, C_6\}$
      where $C_1 = \{-1, -2, 3\}$, $C_2 = \{-3\}$, $C_3 = \{-4, 5\}$, $C_4 = \{-4, -5, 6\}$, $C_5 = \{2, -5, -6\}$, $C_6 = \{4\}$
      ii. $\alpha_2 \stackrel{\text{def}}{=} \{D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8\}$
      where $D_1 = \{2, -3, -5\}$, $D_2 = \{1, -2, -3\}$, $D_3 = \{-1, -5, 7\}$, $D_4 = \{3, -4\}$, $D_5 = \{5, -6\}$, $D_6 = \{4, -5\}$ $D_7 = \{-1, -7\}$ $D_8 = \{6\}$
   (b) Give an example of a formula that does not have an equivalent Horn formula. Give a convincing argument that it is not equivalent to a Horn formula. [Hint: think about the Horn-SAT algorithm. For a satisfiable horn formula, does it always give the same satisfying assignment no matter how the clauses are ordered? **Prove** or **disprove**. Now think about all the possible satisfying assignments of a satisfiable Horn formula. Is there anything special about the one(s) produced by the Horn-SAT algorithm? I am trying to lead you to a special property about the satisfying assignments of a Horn formula, which I'd like you to discover, state, and prove. Once you have done that, if you find a formula that does not have this property then it cannot be equivalent to a Horn formula.]

[1] We will be dealing with CNF formulas a lot, so we will usually use the following simplified notation. A **clause** is a disjunction of literals such as $(\neg p_1 \vee \neg p_2 \vee p_3)$. We will just write clauses as sets of literals, e.g., $C_3 = \{\neg p_1, \neg p_2, p_3\}$. Furthermore, a CNF formula is a conjunction of clauses. We will just write it as a set of clauses. For example, $\alpha_1 = \{C_1, C_2, C_3, C_4, C_5\}$. Even simpler, SAT solvers usually just use the subscript, i.e., 2 instead of p_2, and they use the minus sign instead of "$\neg$", so C_3 would be represented as $\{-1, -2, 3\}$.

4. Assume that $\alpha \to \beta$ is VALID and that var$(\alpha) \cap$ var$(\beta) = \emptyset$. Prove that $\alpha \in$ UNSAT or $\beta \in$ VALID.

5^*. We saw in class that every propositional formula is equivalent to a formula in CNF and a formula in DNF. However, this does not mean that it is computationally easy to put a formula in one of these normal forms. Show that the process is exponential in the worst case. That is, show that there is an exponent $e > 1$ such that for every n, there is a propositional formula F of length less than n such F is in CNF but any equivalent formula in DNF has length greater than e^n.

* Starred problems are meant to be challenging and it is perfectly okay not to get them. For 513 students, they are extra credit.