This is a closed book and notes exam. No computer or calculators of any kind are allowed. Please make sure that your phone is turned off and out of reach. The last sheet is the crib sheet, still in progress.

Please write your answers right on this test, neatly and clearly, using the back if necessary. There are four problems, some of which have sub-parts. Please show your work. Please write your name on the top of every sheet where there is a space after the word “Name”. If you finish early, please read over your solutions and make sure that they are clear and correct. Partial credit will be given for partial solutions. You have 90 minutes. Good luck!

1. [25 pts.] For the following two formulas, $\varphi_{1,i} \in \text{FO}[\Sigma_{\text{RGB}}]$, $i = 1, 2$
   
   (a) Write the meaning of $\varphi_{1,i}$ as an English statement, $E_{1,i}$.
   
   (b) Give a world $A_{1,i} \in \text{WORLD}[\Sigma_{\text{RGB}}]$ such that $A_{1,i} \models \varphi_{1,i}$.
   
   (c) Give a world $B_{1,i} \in \text{WORLD}[\Sigma_{\text{RGB}}]$ such that $B_{1,i} \models \neg \varphi_{1,i}$.

   \begin{itemize}
   \item $\varphi_{1.1} \overset{\text{def}}{=} \forall x \ (r(g(x)) = x \land g(r(x)) = x)$
   \item $\varphi_{1.2} \overset{\text{def}}{=} \forall x \ r(g(x)) = b(x)$.
   \end{itemize}

2. [25 pts.] For the following pairs of natural numbers: $(a_i, b_i)$, $i = 1, 2$, do the following steps:

   (a) Run Euclid’s algorithm and show each iteration, e.g., $a_i = q_1 \ast b_i + r_1$, etc.
   
   (b) What is $\text{gcd}(a_i, b_i)$?
   
   (c) Follow the steps of Euclid’s algorithm backwards to compute $x_i, y_i \in \mathbb{Z}$ s.t. $a_i \ast x_i + b_i \ast y_i = \text{gcd}(a_i, b_i)$.
   
   (d) If $\text{gcd}(a_i, b_i) = 1$, then compute $a_i^{-1} \mod b_i$ and $b_i^{-1} \mod a_i$. Your answers should be greater than 0 and in the first case less than $b_i$ in the second case less than $a_i$.

   Be careful to check – by multiplying and adding your expressions – that each equation you write is correct!

   \begin{align*}
   a_1 &= 101, & b_1 &= 59; \\
   a_2 &= 147, & b_2 &= 132
   \end{align*}
3. [25 pts.] In this problem you will prove the propositional form of Craig’s Interpolation Theorem: If \( \varphi \rightarrow \beta \) is a tautology then there exists an interpolant, \( \gamma \), such that \( \text{var}(\gamma) \subseteq \text{var}(\varphi) \cap \text{var}(\beta) \) and \( \varphi \rightarrow \gamma \) and \( \gamma \rightarrow \beta \) are both tautologies. Here \( \text{var}(\varphi) \) is the set of propositional variables occurring in \( \varphi \).

For an example, consider \( \varphi = (x \leftrightarrow y) \land (z \rightarrow (x \land \neg y)) \) and \( \beta = (z \rightarrow w) \). Then the interpolant is \( \gamma = \neg z \).

You will prove Craig’s Interpolation Theorem by induction on \( |\text{var}(\varphi) - \text{var}(\beta)| \). Craig’s Interpolation Theorem is thus a fact about the natural numbers: \( \forall n \in \mathbb{N} \alpha(n) \).

(a) Write out the definition of \( \alpha(n) \) in a short, clear way using any combination of English and symbols that you choose.

(b) State and prove the base case.

(c) For the inductive case, state your inductive hypothesis.

(d) Now, let \( \varphi \) and \( \beta \) be arbitrary propositional formulas such that \( \varphi \rightarrow \beta \) is a tautology and \( |\text{var}(\varphi) - \text{var}(\beta)| = n_0 + 1 \). State your goal, i.e., what do you need to prove to complete this proof?

(e) Now we have to figure out how to use the indHyp to prove our goal. Here’s a hint: let \( x \in \text{var}(\varphi) - \text{var}(\beta) \). By \( \psi[F/x] \) we mean the formula that results from \( \psi \) by replacing all occurrences of \( x \) by \( F \). For example, \( (x \lor y)[F/x] = (F \lor y) \equiv y \). Let \( \varphi_0 \overset{\text{def}}{=} \varphi[F/x] \) and \( \varphi_1 \overset{\text{def}}{=} \varphi[T/x] \) it follows that \( \varphi_0 \rightarrow \beta \) and \( \varphi_1 \rightarrow \beta \) are both tautologies. Say why?

(f) (This is the only “hard” part of this question and you won’t lose many points for just skipping this part.) Apply the inductive hypothesis to the pair of formulas, \( (\varphi_0 \lor \varphi_1), \beta \), and complete the proof.

4. [25 pts.] Translate the following two English statements, \( E_{4,i} \), \( i = 1, 2 \) into formulas, \( \varphi_{4,i} \in \text{FO}[\Sigma_{\text{RGB}}] \). Then, for each of these,

- If \( \varphi_{4,i} \) is FO-VALID, then give a natural deduction proof of \( \vdash \varphi_{4,i} \).
- If \( \varphi_{4,i} \) is FO-UNSAT, then give a natural deduction proof of \( \vdash \neg \varphi_{4,i} \).
- If \( \varphi_{4,i} \) is neither, then produce two appropriate worlds \( A_{4,i}, B_{4,i} \) such that \( A_{4,i} \models \varphi_{4,i} \) and \( B_{4,i} \models \neg \varphi_{4,i} \).

- \( E_{4,1} \overset{\text{def}}{=} \) “If \( b \) composed with \( r \) is the identity, then \( r \) is 1:1.”
- \( E_{4,2} \overset{\text{def}}{=} \) “If \( g \) is 1:1 then \( g \) is onto.”
Natural Deduction Rules

Proviso for ∀-i and ∃-e: $x_0$ is a “new” variable, i.e., it does not appear in $\varphi$, $\chi$, or any visible assumption.

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$\Sigma_{RGB} = (E^2, R^2, G^2, B^2, S^1, F^1, H^1, V^1; s^0, t^0, e^1, r^1, g^1, b^1)$

Reflexive $\forall x E(x, x)$

Symmetric $\forall xy (E(x, y) \rightarrow E(y, x))$

Transitive $\forall xyz (E(x, y) \land E(y, z) \rightarrow E(x, z))$

AntiSymmetric $\forall xy (E(x, y) \land E(y, x) \rightarrow x = y)$

Trichotomy $\forall xy (E(x, y) \lor E(y, x) \lor x = y)$