In this discussion you will determine a characterization for when an undirected graph is 2-colorable. A graph is 2-colored if each vertex has some color (say H or V) and no two adjacent vertices have the same color:

$$2\text{-colored} \overset{\text{def}}{=} \forall xy ((V(x) \oplus H(x)) \land (E(x, y) \rightarrow (V(x) \oplus V(y)))).$$

For example, $G_2$ is 2-colored, but $G_1$ is not 2-colorable. Why?

1. Briefly argue why in order for $G$ to be 2-colorable it is necessary that $G$ has no odd-length cycles.

In fact, as you will prove today, the absence of odd-length cycles is also a sufficient condition for $G$ to be 2-colorable.

We want to prove this by induction. To get started let’s try to prove $\forall n \alpha_0(n)$, where $\alpha_0(n)$ is

$$\alpha_0(n) \overset{\text{def}}{=} \text{If } G \text{ has } n \text{ vertices and no odd-length cycles then } G \text{ is 2-colorable.}$$

2. Briefly think about doing this inductive proof. The base case is easy. Say how you would start the inductive step, i.e.,

State the indHyp for $\alpha_0$:

Now, you would take an arbitrary $G_0$ which has exactly $n_0 + 1$ vertices and no odd-length cycle. But how can you use indHyp? The problem, is that you would like to take some vertex $v_0$ and color it $V$. But that forces $v_0$’s neighbors to be colored $H$, so we want to use the inductive hypothesis on a more general graph, i.e., one that already has some vertices colored.

To do this, we need a more general inductive hypothesis and thus a more general theorem. We have to figure out when a graph that is already partially 2-colored, can have that partial 2-coloring filled in to make a full 2-coloring.
Prove by induction that $\forall n \alpha(n)$ where $\alpha(n) \overset{\text{def}}{=} \text{If } G \text{ has } n \text{ uncolored vertices, no odd-length cycles, no odd-length paths between vertices of the same color, and no even-length paths between vertices of different colors, Then } G$’s coloring can be extended to a valid 2-coloring.

3. State and prove the base case, $n = 0$.

4. State indHyp:

5. Prove the inductive case.