Lecture 7: Pred Logic = First-Order Logic with Equality + Natural Deduction Proof Rules

\[ \Sigma_g = (E^2, S^1, F^1, H^1, V^1; s, t) \] is the vocabulary of colored graphs.

1 binary predicate symbol, 4 unary predicate symbols, 2 constant symbols

We show the arity superscripts on the symbols only in the definition of the vocabulary. Otherwise they can be deduced from their use in Pred Logic formulas.

The choice of the number and names of the unary color predicates is not important. Any other set of color choices would have a slightly different vocabulary, but would still be called a colored graph.

A colored graph is a world (structure) of vocabulary \( \Sigma_g \). It gives meaning (semantics) to every Pred Logic formula of vocabulary \( \Sigma_g \).

\( s^{G_0} = 4 \), \( G_0 \) interprets the constant symbol \( s \) as the vertex 4.

\[
G_0 = \left( U^{G_0} = \{0, \ldots, 4\}, \right.
\]

\[
E^{G_0} = \{(0, 1), (1, 2), (2, 3), (2, 4), (3, 4), (4, 4)\},
\]

\[
S^{G_0} = \{0\}, \quad F^{G_0} = \{4\}, \quad H^{G_0} = \{3, 4\},
\]

\[
V^{G_0} = \{2, 4\}, \quad s^{G_0} = 4, \quad t^{G_0} = 1\)

Every universe is non-empty. A graph’s universe is its vertices; singular: vertex.
Precedence of Operators: Pred Logic extends and Is Consistent withn Prop Logic

- \( \neg, \forall, \exists \)
- \( \land, \lor, \oplus \)
- \( \rightarrow, \leftrightarrow \)

\( \land, \lor, \text{ and } \oplus \) are associative; \( \rightarrow \) associates from right to left.

\[
\neg a \land b \rightarrow c \equiv ((\neg a) \land b) \rightarrow c
\]
\[
a \rightarrow b \rightarrow c \equiv a \rightarrow (b \rightarrow c)
\]

\[
\forall x \forall y E(x, y) \rightarrow E(y, x) \equiv (\forall x \forall y E(x, y)) \rightarrow E(y, x)
\]
Last Time: Is the following formula true in $G_0$?

$\Rightarrow A:\text{ yes, } G_0 \models \varphi \quad \text{B: no, } G_0 \models \neg \varphi$

$\varphi = \forall x \exists y E(x, y)$

In English: “Every vertex has an outgoing edge.”
Last Time: Is the following formula true in $G_0$?

$\Rightarrow A$: yes, $G_0 \models \varphi$ \quad $B$: no, $G_0 \models \neg \varphi_1$

$\varphi = \forall xy (S(x) \land E(x, y) \to y = t)$

In English: “Any edge out of a start vertex must go to $t$."

![Diagram of $G_0$](image-url)
Clicker Question 7.1  Is the following formula true in $G_0$?

A: yes, $G_0 \models \varphi$    ⇒ B: no, $G_0 \not\models \neg \varphi$

$$\varphi = \forall x \exists y E(y, x)$$

In English: “Every vertex has an incoming edge.”

No, this fails for $x = 0$. 

[Diagram of $G_0$ showing vertices and edges]
Clicker Question 7.2  Is the following formula true in $G_0$?

\[ \Rightarrow \textbf{A: yes, } G_0 \models \varphi \quad \textbf{B: no, } G_0 \models \neg \varphi \]

\[ \varphi = \exists x \ E(x, x) \]

In English: “This graph has a loop.”

Yes, $x = 4$ is a witness.
Clicker Question 7.3  Is the following formula true in $G_0$?

$\Rightarrow$ A: yes, $G_0 \models \varphi$  B: no, $G_0 \models \neg \varphi$

$$\varphi = \exists x \exists y (E(x, t) \land E(t, y))$$

In English: “$t$ has an incoming edge and an outgoing edge.”

Yes, the pair $x = 0, y = 2$ is a witness.
Abbreviations

http://people.cs.umass.edu/~immerman/cs250/definitionsAbbreviations.pdf
Negation of Pred Logic Formulas

\[ \neg \forall x (\alpha) \equiv \exists x (\neg \alpha) \]
\[ \neg \exists x (\alpha) \equiv \forall x (\neg \alpha) \]
\[ \alpha_2 \equiv \forall xy (S(x) \land E(x, y) \rightarrow y = t) \]
\[ \equiv \text{“All edges out of start vertices go to } t\text{.”} \]
\[ \neg \alpha_2 \equiv \exists xy (S(x) \land E(x, y) \land y \neq t) \]
\[ \equiv \text{“Some edge out of a start vertex doesn’t go to } t\text{.”} \]
\[ \alpha_3 \equiv \forall x \exists y E(y, x) \]
\[ \equiv \text{“Every vertex has an incoming edge.”} \]
\[ \neg \alpha_3 \equiv \exists x \forall y \neg E(y, x) \]
\[ \equiv \text{“Some vertex has no incoming edge.”} \]
**Notation: terms**

**Variables:** \( \text{VAR} = \{u, v, w, x, y, z, u_1, v_1, w_1, x_1, y_1, z_1, u_2, v_2, \ldots\} \)

A variable can be temporarily assigned to any element of the universe of a world.

**Constants:** \( c, k, s, t, c_1, k_1, c_2, k_2, \ldots \) \( \text{const}(\Sigma_g) = \{s, t\} \)

A constant is a permanent name for a certain element of the universe of a world.

**Terms:** \( t_1, t_2, \ldots \) \( \text{term}(\Sigma_g) = \{s, t\} \cup \text{VAR} \)

**Def:** A term is an expression denoting an element of the universe.

Terms are: variables, or constants, or functions of other terms.
Free and Bound Variables

An occurrence of a variable $x$ is **bound** iff it occurs within the scope of a quantifier, $\forall x$ or $\exists x$. Otherwise the occurrence is **free**.

\[ \exists y z (y \neq z \land E(x, y) \land E(x, z)) \quad \text{“} x \text{ has at least two outgoing edges.”} \]

\[ \forall z (z + x = z) \quad \text{“} x \text{ is the identity for } +.\text{”} \]

\[ \forall y (y + x = y) \quad \text{“} x \text{ is the identity for } +.\text{”} \]

\[ \forall x (x + x = x) \quad \text{“Addition is idempotent.”} \]

\[ x \neq y \land \exists y (y < x) \quad \text{“} x, y \text{ are distinct and } x \text{ is not minimum.”} \]

\[ x \neq y \land \exists z (z < x) \quad \text{“} x, y \text{ are distinct and } x \text{ is not minimum.”} \]

Bound variables are dummy variables.

A Pred Logic formula says something about its free variables.
Substitution of Variables

\[ \varphi[t/x] \overset{\text{def}}{=} (\text{free occurrences in } \varphi \text{ of variable } x) \leadsto (\text{term } t) \]

\[ \alpha \equiv \forall y(y + x = y) \quad \text{“}x\text{ is identity for +.”} \]

\[ \alpha[0/x] \equiv \forall y(y + 0 = y) \quad \text{“}0\text{ is identity for +.”} \]

\[ \alpha[z/x] \equiv \forall y(y + z = y) \quad \text{“}z\text{ is identity for +.”} \]

\[ \alpha[(z + 2)/x] \equiv \forall y(y + (z + 2) = y) \quad \text{“(}z + 2\text{) is identity for +.”} \]

\[ \alpha[y/x] \equiv \forall y_1(y_1 + y = y_1) \quad \text{“}y\text{ is identity for +.”} \]

Bound variables are automatically changed to avoid capture of variables in substituted terms.
Natural Deduction Proof Rules