CS250: Discrete Math for Computer Science

L7: Predicate Calculus: PredCalc

PredCalc $=$ First-Order Logic with Equality

Generalization of PropCalc:

rich enough to express all of mathematics!

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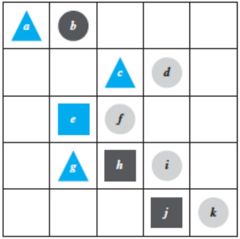
Let's start by reviewing the R7 Quiz.

$T_1 \quad =$



1. $T_1 \models \forall x(\text{Triangle}(x) \to \text{Blue}(x))$ **True:** all triangles are blue.
2. $T_1 \models \forall x(\text{Blue}(x) \to \text{Triangle}(x))$ **False:** *e* is a Blue Square.
3. $T_1 \models \forall x(\text{Square}(x) \to \exists y \text{LeftOf}(y, x))$ **True:** *a* is LeftOf all the Squares.
4. $T_1 \models \forall x(\text{Gray}(x) \to \text{Circle}(x))$ **True:** All Gray elements are Circles.
5. $T_1 \models \forall x(\text{Circle}(x) \to \text{Gray}(x))$ **False:** Not all Circles are Gray.
6. $T_1 \models \exists x \forall y(\text{LeftOf}(y, x) \vee y = x)$ **True:** there is a unique rightmost element.

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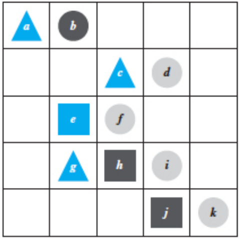
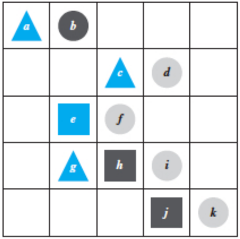


7. $T_1 \models \exists x \forall y (\text{Above}(y, x) \lor y = x)$
   False: there is no unique bottom most element.

8. $T_1 \models \forall x \exists y (\text{Triangle}(x) \rightarrow \text{LeftOf}(x, y) \land \text{Circle}(y))$
   True: every Triangle has a Circle to its right.

9. $T_1 \models \forall x \exists y (\text{Triangle}(x) \rightarrow \sim\text{Above}(x, y) \land \sim\text{Above}(y, x) \land \text{Circle}(y))$
   True: Every Triangle shares its row with a circle.

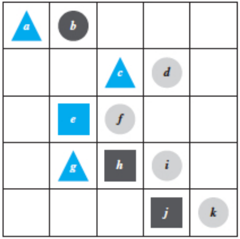
10. $T_1 \models \exists x \exists y (\text{Square}(x) \land \text{Circle}(y) \land Above(y, x) \land \text{Leftof}(x, y))$
    True: witnesses are $x := h$ and $y := d$.

$T_1 \quad =$ 

iClicker 7.1 True or False:

$$T_1 \models \forall xy\, (\text{Gray}(x) \land \text{LeftOf}(x, y) \to \sim \text{Blue}(y)) \ ?$$

A: True **B: False**

$T_1 \quad =$ 

iClicker 7.2 True or False:

$$T_1 \models \forall x \exists y\, (\text{Circle}(y) \wedge \sim\text{Above}(x, y) \wedge \sim\text{Above}(y, x))\ ?$$

A: True **B: False**

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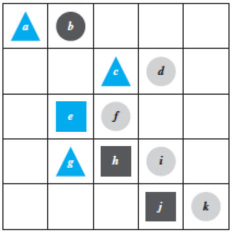

Tarski's World vocabulary: $\Sigma_{\text{Tarski}} =$

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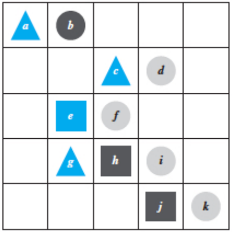

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Σ_{Tarski} has six **predicate symbols** of **arity** 1, and two of arity 2.

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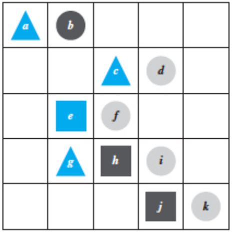

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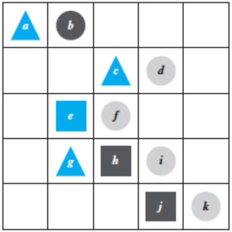

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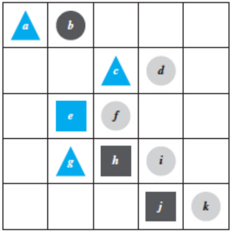

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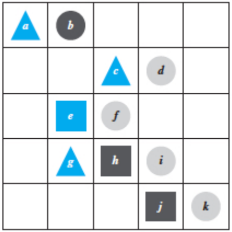

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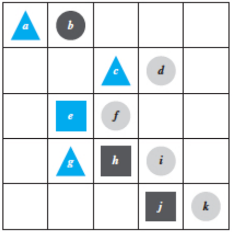

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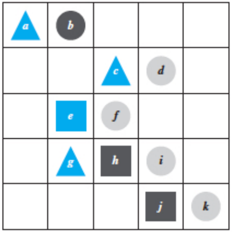

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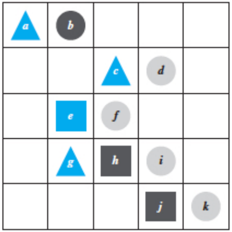

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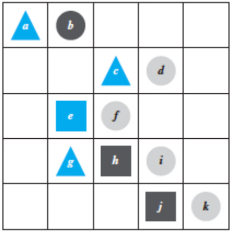

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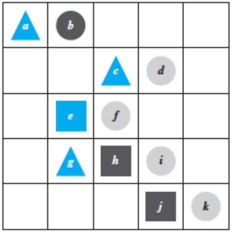

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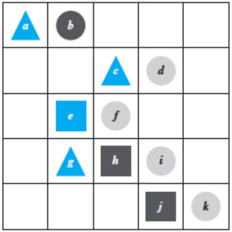

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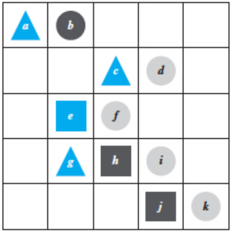

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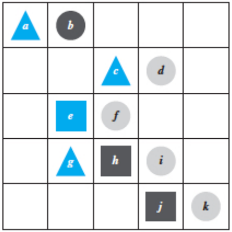

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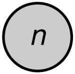

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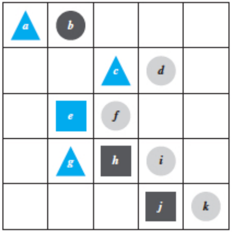

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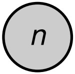

$$
\begin{aligned}
|N_1| &= \{n\}\\
\text{Circle}^{N_1} &= \{n\}\\
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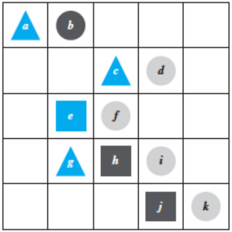
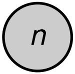


Exercise: Which sentences of the R7 quiz

$N_1 =$



does Nishal's world, N_1, satisfy?

$$
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