CS250: Discrete Math for Computer Science

L6: CNF and Natural Deduction for PropCalc
How to Simplify a PropCalc Formula:

\[(p \rightarrow q) \rightarrow ((q \rightarrow r) \land p) \equiv\]
How to Simplify a PropCalc Formula:

1. Get rid of $\rightarrow$’s using \textbf{def. of implication}.

\[(p \rightarrow q) \rightarrow ((q \rightarrow r) \land p) \equiv (\sim p \lor q) \rightarrow ((\sim q \lor r) \land p)\]
\[\equiv \sim ((\sim p \lor q) \lor ((\sim q \lor r) \land p))\]
How to Simplify a PropCalc Formula:

1. Get rid of $\to$’s using def. of implication.
2. Push $\neg$’s all the way inside using de Morgan.
   Now its in Negation Normal Form (NNF).

\[(p \to q) \to ((q \to r) \land p) \equiv (\neg p \lor q) \to ((\neg q \lor r) \land p)\]
\[\equiv \neg (\neg p \lor q) \lor ((\neg q \lor r) \land p)\]
\[\text{NNF} \equiv (p \land \neg q) \lor ((\neg q \lor r) \land p)\]
How to Simplify a PropCalc Formula:

1. Get rid of \( \rightarrow \)'s using **def. of implication**.
2. Push \( \sim \)'s all the way inside using **de Morgan**.
   Now its in Negation Normal Form (NNF).
3. Use distributive, associative, commutative to put in either
   **Disjunctive Normal Form (DNF): \( \lor \)'s of \( \land \)'s of literals**

   A **literal** is a prop variable or its negation, e.g., \( p, \sim q \).

\[
(p \rightarrow q) \rightarrow ((q \rightarrow r) \land p) \equiv \sim(p \lor q) \rightarrow ((\sim q \lor r) \land p) \\
\equiv \sim(\sim(p \lor q) \lor ((\sim q \lor r) \land p)) \\
\text{NNF} \equiv (p \land \sim q) \lor ((\sim q \lor r) \land p) \\
\text{NNF} \equiv (p \land \sim q) \lor ((\sim q \land p) \lor (r \land p)) \\
\text{DNF} \equiv (p \land \sim q) \lor (p \land r)
\]
How to Simplify a PropCalc Formula:

1. Get rid of \( \rightarrow \)'s using **def. of implication**.
2. Push \( \sim \)'s all the way inside using **de Morgan**.
   Now its in Negation Normal Form (**NNF**).
3. Use distributive, associative, commutative to put in either
   **Disjunctive Normal Form** (**DNF**: \( \lor \)'s of \( \land \)'s of literals)
   **Conjunctive Normal Form** (**CNF**: \( \land \)'s of \( \lor \)'s of literals)

   A **literal** is a prop variable or its negation, e.g., \( p, \sim q \).

\[
(p \to q) \to ((q \to r) \land p) \equiv (\sim p \lor q) \to ((\sim q \lor r) \land p) \\
\equiv \sim (\sim p \lor q) \lor ((\sim q \lor r) \land p) \\
\equiv ((p \land \sim q) \lor ((\sim q \lor r) \land p)) \\
\equiv (p \land \sim q) \lor (p \land r) \\
\equiv p \land (\sim q \lor r)
\]
Which of the following formulas is in CNF (conjunction of disjunctions of literals)?

A: \((p \land \sim q \land r) \lor (\sim p \land q \land s) \lor (q \land r \land \sim s)\)

B: \((\sim p \lor q \lor \sim r) \land (p \lor \sim q \lor \sim s) \land (\sim q \lor \sim r \lor s)\)

C: \((p \land \sim q) \lor ((\sim q \lor p) \land (r \lor p))\)

D: \(\sim (\sim p \lor q) \lor ((\sim q \lor r) \land p)\)
R6: Our PropCalc proof rules are slightly different from Epp’s proof rules.

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<tr>
<td>∧</td>
<td>[ \frac{p \quad q}{p \land q} ]</td>
<td>[ \frac{p \land q}{p} \quad \frac{p \land q}{q} ]</td>
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<td>[ \frac{p \rightarrow F}{\sim p} \quad \frac{\sim p \rightarrow F}{p} ]</td>
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<td>∼∼</td>
<td>[ \frac{p}{\sim \sim p} ]</td>
<td>[ \frac{\sim \sim p}{p} ]</td>
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Natural Deduction rule: →-introduction

\[
\begin{array}{c|c|c}
1 & p & \\ \\
2 & \ & \ \ \\
3 & q & \ \ \\
4 & p \rightarrow q & \rightarrow\text{-i, 1–3} \\
\end{array}
\]

\[\frac{p \vdash q}{p \rightarrow q}\]
Natural Deduction rule: →-introduction

1 \[ p \]

2

3 \[ q \]

4 \[ p \rightarrow q \] →-i, 1–3

Notation: \( p \vdash q \) (\( p \) proves \( q \)): From assumption \( p \), can prove \( q \).
Natural Deduction rule: \( \rightarrow \text{-introduction} \)

1
\[ p \]

2
\[ q \]

3
\[ p \rightarrow q \]

4
\[ p \vdash q \quad \rightarrow \text{-i, 1–3} \]

\[ \frac{p \vdash q}{p \rightarrow q} \]

\textbf{Notation:} \( p \vdash q \) (\( p \) proves \( q \)): From assumption \( p \), can prove \( q \).

\textbf{Proposition:} \( \rightarrow \text{-i} \) is \textbf{sound}, i.e., if from assumption \( p \) we can prove \( q \), then every world satisfies \( p \rightarrow q \).
Natural Deduction rule: \(\rightarrow\) introduction

\[
\begin{array}{c|c}
1 & p \\
2 & \\
3 & q \\
4 & p \rightarrow q & \rightarrow\text{-}i, 1–3
\end{array}
\]

\[\frac{p \vdash q}{p \rightarrow q}\]

**Notation:** \(p \vdash q\) (**\(p\) proves \(q\)**): From assumption \(p\), can prove \(q\).

**Proposition:** \(\rightarrow\)-i is **sound**, i.e., if from assumption \(p\) we can prove \(q\), then every world satisfies \(p \rightarrow q\).

**Proof.**
Since \(p \vdash q\), and by the soundness of the proof rules so far, we know that every world that satisfies \(p\) must also satisfy \(q\). Thus every world satisfies \(p \rightarrow q\). \(\square\)
Natural Deduction rule: →-introduction

\[
\begin{array}{c|c}
1 & p \\
2 & \\
3 & q \\
4 & p \rightarrow q \quad \rightarrow\text{-i, 1–3} \\
\end{array}
\]

\[\frac{p \vdash q}{p \rightarrow q}\]

Notation: \( p \vdash q \) (\( p \) proves \( q \)): From assumption \( p \), can prove \( q \).

Proposition: \( \rightarrow\text{-i} \) is sound, i.e., if from assumption \( p \) we can prove \( q \), then every world satisfies \( p \rightarrow q \).

Proof.
Since \( p \vdash q \), and by the soundness of the proof rules so far, we know that every world that satisfies \( p \) must also satisfy \( q \). Thus every world satisfies \( p \rightarrow q \). \( \square \)

More about this once we have studied inductive proofs.
Example use of →-introduction

1 | p
2 | r ∨ p  \( \lor \)-i, 1
3 | p → (r ∨ p)  →-i, 1–2
Example use of $\rightarrow$-introduction

Thus, $\vdash p \rightarrow (r \lor p)$
Natural Deduction rule: **F-e**  Proof by Contradiction

\[ \begin{array}{l}
\hline
1 & \sim p \\
2 & F \\
3 & F \\
4 & p & F-e, 1–3 \\
\hline
\end{array} \]

\[ \frac{\sim p}{p} \]
\[ \frac{p}{\sim p} \]

Proposition: F-e is sound, i.e., if \( \sim p \vdash F \) then every world satisfies \( p \).

Proof.

Since \( \sim p \vdash F \), by the soundness of the proof rules so far, we know that every world that satisfies \( \sim p \) must also satisfy \( F \). But no world satisfies \( F \). Thus every world satisfies \( p \).

\( \square \)
Natural Deduction rule: \( F-e \)   Proof by Contradiction

1  \[ \sim p \]
2  \[ \]  \[ \]  \[ \sim p \vdash F \]  \[ p \vdash F \]
3  \[ F \]  \[ p \]  \[ \sim p \]
4  \[ p \]  \[ F-e, 1-3 \]

**Proposition:** \( F-e \) is **sound**, i.e., if \( \sim p \vdash F \) then every world satisfies \( p \).
Natural Deduction rule: F-e  

Proof by Contradiction

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<td>1</td>
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</tr>
<tr>
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<td></td>
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<tr>
<td>3</td>
<td></td>
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<tr>
<td>4</td>
<td>p</td>
<td>F-e, 1–3</td>
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</table>

Proposition: F-e is **sound**, i.e., if ∼p ⊨ F then every world satisfies p.

**Proof.**

Since ∼p ⊨ F, by the soundness of the proof rules so far, we know that every world that satisfies ∼p must also satisfy F. But no world satisfies F. Thus every world satisfies p. □
Proposition: $\lor$-e is sound.

Proof. Since $p \vdash r$ and $q \vdash r$, every world that satisfies $p$ or satisfies $q$ satisfies $r$. Thus every world that satisfies $p \lor q$ satisfies $r$. □
Natural Deduction rule: ∨-e

\[ \begin{align*}
1 & \quad p \lor q \\
2 & \quad p \\
3 & \quad r \\
4 & \quad r \\
5 & \quad q \\
6 & \quad q \\
7 & \quad r \\
8 & \quad r
\end{align*} \]

\[ \begin{align*}
\frac{p \lor q \quad p \vdash r \quad q \vdash r}{r}
\end{align*} \]

Proposition: ∨-e is **sound**.
Proposition: $\lor$-e is sound.

Proof.

Since $p \vdash r$ and $q \vdash r$, every world that satisfies $p$ or satisfies $q$ satisfies $r$. Thus every world that satisfies $p \lor q$ satisfies $r$. $\square$
What is the justification for line 4?

A: $\land$-i
B: $\land$-e
C: $\lor$-i
D: $\lor$-e

What is the justification for line 8?

A: $\land$-i
B: $\land$-e
C: F-i
D: F-e
\begin{align*}
1 & \quad \sim p \lor \sim q \\
2 & \quad p \land q \\
3 & \quad \sim p \\
4 & \quad p, 2 \\
5 & \quad F, F-i, 3, 4 \\
6 & \quad \sim q \\
7 & \quad q, \land-e, 2 \\
8 & \quad F, 6, 7 \\
9 & \quad F, \lor-e, 1, 3–5, 6–8 \\
10 & \quad \sim (p \land q), F-e, 2–9
\end{align*}

\textbf{iClicker 6.2} What is the justification for line 4?

A: \quad \land-i  \\
B: \quad \land-e  \\
C: \quad \lor-i  \\
D: \quad \lor-e

\textbf{iClicker 6.3} What is the justification for line 8?

A: \quad \land-i  \\
B: \quad \land-e  \\
C: \quad F-i  \\
D: \quad F-e
1 \( \sim p \lor \sim q \)

2 \( p \land q \)

3 \( \sim p \)

4 \( p, 2 \)

5 \( \text{F} \) \( \text{F-i, 3, 4} \)

6 \( \sim q \)

7 \( q \) \( \land-e, 2 \)

8 \( \text{F} \) \( , 6, 7 \)

9 \( \text{F} \) \( \lor-e, 1, 3-5, 6-8 \)

10 \( \sim (p \land q) \) \( \text{F-e, 2-9} \)

**iClicker 6.2** What is the justification for line 4?

A: \( \land-i \)
B: \( \land-e \)
C: \( \lor-i \)
D: \( \lor-e \)

**iClicker 6.3** What is the justification for line 8?

A: \( \land-i \)
B: \( \land-e \)
C: \( \text{F-i} \)
D: \( \text{F-e} \)
R6: Our PropCalc proof rules are slightly different from Epp’s proof rules.

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<td>$\sim\sim p$</td>
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Propositional Equivalence

PropCalc formulas $p$ and $q$ are equivalent ($p \equiv q$) iff they agree on every row of their truth tables.

Observation: $p \equiv q$ iff $p \leftrightarrow q$ is a tautology.

Some Important Equivalences (worth memorizing):

- **double negation**
  $$p \equiv \sim\sim p$$

- **de Morgan**
  $$\sim(p \lor q) \equiv \sim p \land \sim q$$

- **de Morgan**
  $$\sim(p \land q) \equiv \sim p \lor \sim q$$

- **def. of implication**
  $$p \rightarrow q \equiv \sim p \lor q$$

- **contrapositive**
  $$p \rightarrow q \equiv \sim q \rightarrow \sim p$$

- **def. of iff**
  $$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$$
Propositional Equivalence

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**Some Important Equivalences (worth memorizing):**

- **double negation**
  
  $p \equiv \neg\neg p$


Propositional Equivalence

PropCalc formulas \( p \) and \( q \) are equivalent \((p \equiv q)\) iff they agree on every row of their truth tables.

**Observation:** \( p \equiv q \) iff \( p \leftrightarrow q \) is a tautology.

**Some Important Equivalences (worth memorizing):**

- **double negation** \( p \equiv \sim \sim p \)
- **de Morgan** \( \sim (p \lor q) \equiv \sim p \land \sim q \)
- **de Morgan** \( \sim (p \land q) \equiv \sim p \lor \sim q \)
Propositional Equivalence

PropCalc formulas $p$ and $q$ are equivalent ($p ≡ q$) iff they agree on every row of their truth tables.

**Observation:** $p ≡ q$ iff $p \leftrightarrow q$ is a tautology.

**Some Important Equivalences (worth memorizing):**

- **double negation**
  
  $p \equiv \sim \sim p$

- **de Morgan**
  
  $\sim (p \lor q) \equiv \sim p \land \sim q$

- **de Morgan**
  
  $\sim (p \land q) \equiv \sim p \lor \sim q$

- **def. of implication**
  
  $p \to q \equiv \sim p \lor q$
Propositional Equivalence

PropCalc formulas $p$ and $q$ are equivalent ($p \equiv q$) iff they agree on every row of their truth tables.

Observation: $p \equiv q$ iff $p \leftrightarrow q$ is a tautology.

Some Important Equivalences (worth memorizing):

- **double negation**
  \[ p \equiv \neg\neg p \]

- **de Morgan**
  \[ \neg(p \lor q) \equiv \neg p \land \neg q \]

- **de Morgan**
  \[ \neg(p \land q) \equiv \neg p \lor \neg q \]

- **def. of implication**
  \[ p \rightarrow q \equiv \neg p \lor q \]

- **contrapositive**
  \[ p \rightarrow q \equiv \neg q \rightarrow \neg p \]
Propositional Equivalence

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**Some Important Equivalences (worth memorizing):**

- **double negation**
  \[ p \equiv \sim \sim p \]

- **de Morgan**
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- **de Morgan**
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- **def. of implication**
  \[ p \rightarrow q \equiv \sim p \lor q \]

- **contrapositive**
  \[ p \rightarrow q \equiv \sim q \rightarrow \sim p \]

- **def. of iff**
  \[ p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p) \]
More Important Equivalences (worth memorizing):

- **commutative**
  \[ p \lor q \equiv q \lor p \]

- **commutative**
  \[ p \land q \equiv q \land p \]
More Important Equivalences (worth memorizing):

- Commutative: \( p \lor q \equiv q \lor p \)
- Commutative: \( p \land q \equiv q \land p \)
- Associative: \( p \land (q \land r) \equiv (p \land q) \land r \)
- Associative: \( p \lor (q \lor r) \equiv (p \lor q) \lor r \)

- Distributive
  - \( p \lor (q \land r) \equiv (p \lor q) \land (p \lor r) \)
  - \( p \land (q \lor r) \equiv (p \land q) \lor (p \land r) \)

- Excluded Middle: \( p \lor \neg p \equiv T \)
More Important Equivalences (worth memorizing):

- **commutative**  
  \[ p \lor q \equiv q \lor p \]

- **commutative**  
  \[ p \land q \equiv q \land p \]

- **associative**  
  \[ p \land (q \land r) \equiv (p \land q) \land r \]

- **associative**  
  \[ p \lor (q \lor r) \equiv (p \lor q) \lor r \]

- **distributive**  
  \[ p \lor (q \land r) \equiv (p \lor q) \land (p \lor r) \]

- **distributive**  
  \[ p \land (q \lor r) \equiv (p \land q) \lor (p \land r) \]
More Important Equivalences (worth memorizing):

- **commutative** $p \lor q \equiv q \lor p$
- **commutative** $p \land q \equiv q \land p$
- **associative** $p \land (q \land r) \equiv (p \land q) \land r$
- **associative** $p \lor (q \lor r) \equiv (p \lor q) \lor r$
- **distributive** $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$
- **distributive** $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
- **excluded middle** $p \lor \sim p \equiv T$
SAT is an important class.

If formula $a$ has $n$ PVARs, $p_1, \ldots, p_n$, it would seem to require about $2^n$ time in the worst case to test if $a \in \text{SAT}$. 
SAT is an important class.

If formula $a$ has $n$ PVARs, $p_1, \ldots, p_n$, it would seem to require about $2^n$ time in the worst case to test if $a \in \text{SAT}$.

But, if we are given a satisfying world $W$ for $a$, we can check immediately that $W \models a$. 
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This sort of search problem: exponentially many possibilities, each one easy to verify, corresponds to Nondeterministic Polynomial Time (NP).
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But, if we are given a satisfying world $W$ for $a$, we can check immediately that $W \models a$.

This sort of search problem: exponentially many possibilities, each one easy to verify, corresponds to Nondeterministic Polynomial Time (NP).

In fact, $\text{SAT}$ is a hardest such problem: NP complete.

You will learn much more about this in CMPSCI 311.
Knights always truthful;  Knaves always lie;  \( A, B \in \{\text{Kt,Kv}\} \)

\( B : \text{“A&B opposite types”} \)
Knights always truthful; Knaves always lie; \( A, B \in \{\text{Kt,Kv}\} \)

\[
\begin{align*}
A & : \text{“}B\text{ is Kt”} \\
B & : \text{“}A\&B\text{ opposite types”}
\end{align*}
\]
Knights always truthful;  Knaves always lie;  \( A, B \in \{\text{Kt}, \text{Kv}\} \)

\[ A : \text{“B is Kt”} \quad \quad B : \text{“A&B opposite types”} \]

\[ T_1 \overset{\text{def}}{=} B \text{ is a Kt} \quad \quad T_2 \overset{\text{def}}{=} A \& B \text{ opposite types} \]
Knights and Knaves

[Smullyan, What Is the Name of This Book?]

Knights always truthful; Knaves always lie; \( A, B \in \{\text{Kt,Kv}\} \)

\[ S_1 \overset{\text{def}}{=} A : \text{“B is Kt”} \quad S_2 \overset{\text{def}}{=} B : \text{“A&B opposite types”} \]

\[ T_1 \overset{\text{def}}{=} B \text{ is a Kt} \quad T_2 \overset{\text{def}}{=} A&B \text{ opposite types} \]
Knights always truthful; Knaves always lie; \( A, B \in \{ \text{Kt,Kv} \} \)

\[
S_1 \overset{\text{def}}{=} A : \text{“B is Kt”} \quad S_2 \overset{\text{def}}{=} B : \text{“A&B opposite types”}
\]

\[
T_1 \overset{\text{def}}{=} B \text{ is a Kt} \quad T_2 \overset{\text{def}}{=} A&B \text{ opposite types}
\]

\[
S_1 = T_1 \leftrightarrow A \text{ is Kt} \quad S_2 = T_2 \leftrightarrow B \text{ is Kt}
\]
Knights and Knaves

Knights always truthful;  Knaves always lie;  \(A, B \in \{\text{Kt}, \text{Kv}\}\)

\[
S_1 \overset{\text{def}}{=} A : "B \text{ is Kt}"
\]

\[
S_2 \overset{\text{def}}{=} B : "A&B \text{ opposite types}"
\]

\[
T_1 \overset{\text{def}}{=} B \text{ is a Kt}
\]

\[
T_2 \overset{\text{def}}{=} A&B \text{ opposite types}
\]

\[
S_1 = T_1 \leftrightarrow A \text{ is Kt}
\]

\[
S_2 = T_2 \leftrightarrow B \text{ is Kt}
\]

<table>
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<th>W</th>
<th>A is Kt</th>
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<th>(T_2)</th>
<th>(T_1 \leftrightarrow A \text{ is Kt})</th>
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<tr>
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</tr>
</tbody>
</table>

\(W_0\) is only world satisfying \(S_1 \land S_2\).

Thus \(A\) and \(B\) are both Knaves.
Knights and Knaves  

[Smullyan, *What Is the Name of This Book?*]

Knights always truthful; Knaves always lie; \( A, B \in \{\text{Kt, Kv}\} \)

\[
S_1 \overset{\text{def}}{=} \text{“} B \text{ is Kt”} \quad S_2 \overset{\text{def}}{=} \text{“} A \& B \text{ opposite types”} \\
T_1 \overset{\text{def}}{=} \text{“} B \text{ is a Kt} \quad T_2 \overset{\text{def}}{=} \text{“} A \& B \text{ opposite types} \\
S_1 = T_1 \iff A \text{ is Kt} \quad S_2 = T_2 \iff B \text{ is Kt}
\]

<table>
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<th>( W )</th>
<th>( A \text{ is Kt} )</th>
<th>( B \text{ is Kt} )</th>
<th>( T_1 )</th>
<th>( T_2 )</th>
<th>( T_1 \iff A \text{ is Kt} )</th>
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</tbody>
</table>

\( W_0 \) is only world satisfying \( S_1 \land S_2 \).
Knights and Knaves  [Smullyan, *What Is the Name of This Book?*

Knights always truthful;  Knaves always lie;  \(A, B \in \{\text{Kt}, \text{Kv}\}\)

\[
S_1 \overset{\text{def}}{=} A : \text{"B is Kt"} \quad S_2 \overset{\text{def}}{=} B : \text{"A&\&B opposite types"}
\]

\[
T_1 \overset{\text{def}}{=} \text{B is a Kt} \quad T_2 \overset{\text{def}}{=} \text{A\&\&B opposite types}
\]

\[
S_1 = T_1 \leftrightarrow A \text{ is Kt} \quad S_2 = T_2 \leftrightarrow B \text{ is Kt}
\]

<table>
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<th>W</th>
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<th>B is Kt</th>
<th>T_1</th>
<th>T_2</th>
<th>T_1 \leftrightarrow A is Kt</th>
<th>T_2 \leftrightarrow B is Kt</th>
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<tr>
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<td>1</td>
<td>0</td>
<td>1</td>
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<tr>
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</tbody>
</table>

\(W_0\) is only world satisfying \(S_1 \land S_2\).

**Thus A and B are both Knaves.**
R6 Quiz Answers: Match the Epp Proof Rules to the equivalent Natural Deduction Rules.

1. Modus Ponens: $\rightarrow$-e
2. Modus Tollens: $\rightarrow$-e
3. Generalization: $\lor$-i
4. Specialization: $\land$-e
5. Conjunction: $\land$-i

6, 7. In the following proof, identify the natural deduction rules used in lines 2 and 3.

```
1  p \land q
2  q \quad \land$-e, 1
3  (\sim r \lor q) \quad \lor$-i, 2
```
8. Is the following a sound proof rule?

\[ p \rightarrow q \quad q \quad p \]

In answering this, you may consider the worlds shown in this truth table:

<table>
<thead>
<tr>
<th>( W )</th>
<th>( p )</th>
<th>( q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_3 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( W_2 )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( W_1 )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( W_0 )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Not valid: reasoning from the converse fails in world \( W_1 \).

9. In Smullyan’s Island of Knights and Knaves, two natives C and D approach you but only C speaks. C says: Both of us are knaves. What are C and D? C is a Knave and D is a Knight.

10. In Smullyan’s Island of Knights and Knaves, you encounter natives E and F. E says: F is a knave. F says: E is a knave. How many knaves are there? 1