CS250: Discrete Math for Computer Science

L34: Turing Machines & Unsolvability of Halting

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

By A. M. Turing.

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The "computable" numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means. Although the subject of this paper is ostensibly the computable *numbers*, it is almost equally easy to define and investigate computable functions of an integral variable or a real or computable variable, computable predicates, and so forth. The fundamental problems involved are, however, the same in each case, and I have chosen the computable numbers for explicit treatment as involving the least cumbrous technique. I hope shortly to give an account of the relations of the computable numbers, functions, and so forth to one another. This will include a development of the theory of functions of a real variable expressed in terms of computable numbers. According to my definition, a number is computable if its decimal can be written down by a machine.

In §§ 9, 10 I give some arguments with the intention of showing that the computable numbers include all numbers which could naturally be regarded as computable. In particular, I show that certain large classes of numbers are computable. They include, for instance, the real parts of all algebraic numbers, the real parts of the zeros of the Bessel functions, the numbers π, e, etc. The computable numbers do not, however, include all definable numbers, and an example is given of a definable number which is not computable.

Although the class of computable numbers is so great, and in many ways similar to the class of real numbers, it is nevertheless enumerable. In § 8 I examine certain arguments which would seem to prove the contrary. By the correct application of one of these arguments, conclusions are reached which are superficially similar to those of Gödel†. These results

Turing Machine: $M = (Q, \Sigma, \delta, s)$

Q: finite set of states; start state $s \in Q$

Σ: finite set of symbols, e.g., $\Sigma = \{\triangleright, \sqcup, 0, 1\}$

δ: $Q \times \Sigma \to (Q \cup \{h\}) \times \Sigma \times \{\leftarrow, \rightarrow, -\}$

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mvRt.tm	s	q	q_0	q_1
0	$s, 0, \rightarrow$	$q_0, \sqcup, \rightarrow$		
1	$s, 1, \rightarrow$	$q_1, \sqcup, \rightarrow$		
$\sqcup$	$q, \sqcup, \leftarrow$		$s, 0, \leftarrow$	$s, 1, \leftarrow$
$\triangleright$	$s, \triangleright, \rightarrow$	$h, \triangleright, -$		
comment	find $\sqcup$	memorize & erase	change $\sqcup$ to 0	change $\sqcup$ to 1

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$\sqcup$	$q, \sqcup, \leftarrow$		$s, 0, \leftarrow$	$s, 1, \leftarrow$
$\triangleright$	$s, \triangleright, \rightarrow$	$h, \triangleright, -$		
comment	find $\sqcup$	memorize & erase	change $\sqcup$ to 0	change $\sqcup$ to 1

s

$\boxed{\triangleright}$	1	1	0	1	$\sqcup$	$\sqcup$	$\cdots$

mvRt.tm		
	s	q
0	$s, 0, \rightarrow$	$q_0, \sqcup, \rightarrow$
1	$s, 1, \rightarrow$	$q_1, \sqcup, \rightarrow$
$\sqcup$	$q, \sqcup, \leftarrow$	
$\triangleright$	$s, \triangleright, \rightarrow$	$h, \triangleright, -$

	q_0	q_1
0		
1		
$\sqcup$	$s, 0, \leftarrow$	$s, 1, \leftarrow$
$\triangleright$		

s	$\boxed{\triangleright}$	1	1	0	1	$\sqcup$	$\sqcup$	$\cdots$

mvRt.tm		
	s	q
0	$s, 0, \rightarrow$	$q_0, \sqcup, \rightarrow$
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	q_0	q_1
0		
1		
$\sqcup$	$s, 0, \leftarrow$	$s, 1, \leftarrow$
$\triangleright$		

s	$\boxed{\triangleright}$	1	1	0	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	$\boxed{\mathbf{1}}$	1	0	1	$\sqcup$	$\sqcup$	$\cdots$

mvRt.tm		
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s	$\boxed{\triangleright}$	1	1	0	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	$\boxed{\mathbf{1}}$	1	0	1	$\sqcup$	$\sqcup$	$\cdots$
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1		
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s	$\boxed{\triangleright}$	1	1	0	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	$\boxed{\mathbf{1}}$	1	0	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	$\boxed{\mathbf{1}}$	0	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	$\boxed{\mathbf{0}}$	1	$\sqcup$	$\sqcup$	$\cdots$

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s	$\triangleright$	$\boxed{\mathbf{1}}$	1	0	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	$\boxed{\mathbf{1}}$	0	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	$\boxed{\mathbf{0}}$	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	0	$\boxed{\mathbf{1}}$	$\sqcup$	$\sqcup$	$\cdots$

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s	$\triangleright$	$\boxed{\mathbf{1}}$	1	0	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	$\boxed{\mathbf{1}}$	0	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	$\boxed{\mathbf{0}}$	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	0	$\boxed{\mathbf{1}}$	$\sqcup$	$\sqcup$	$\cdots$
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s	$\triangleright$	$\boxed{1}$	1	0	1	$\sqcup$	$\sqcup$	..
s	$\triangleright$	1	$\boxed{1}$	0	1	$\sqcup$	$\sqcup$	..
s	$\triangleright$	1	1	$\boxed{0}$	1	$\sqcup$	$\sqcup$	..
s	$\triangleright$	1	1	0	$\boxed{1}$	$\sqcup$	$\sqcup$	..
s	$\triangleright$	1	1	0	1	$\boxed{\sqcup}$	$\sqcup$	..
q	$\triangleright$	1	1	0	$\boxed{1}$	$\sqcup$	$\sqcup$	..

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s	$\triangleright$	$\boxed{\mathbf{1}}$	1	0	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	$\boxed{\mathbf{1}}$	0	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	$\boxed{\mathbf{0}}$	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	0	$\boxed{\mathbf{1}}$	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	0	1	$\boxed{\sqcup}$	$\sqcup$	$\cdots$
q	$\triangleright$	1	1	0	$\boxed{\mathbf{1}}$	$\sqcup$	$\sqcup$	$\cdots$
q_1	$\triangleright$	1	1	0	$\sqcup$	$\boxed{\sqcup}$	$\sqcup$	$\cdots$

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$\vdots$				$\vdots$		$\vdots$		$\vdots$
s	$\triangleright$	1	1	0	1	$\boxed{\sqcup}$	$\sqcup$	$\cdots$
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s	$\triangleright$	1	1	0	$\boxed{\sqcup}$	1	$\sqcup$	$\cdots$
q	$\triangleright$	1	1	$\boxed{\mathbf{0}}$	$\sqcup$	1	$\sqcup$	$\cdots$
q_0	$\triangleright$	1	1	$\sqcup$	$\boxed{\sqcup}$	1	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	$\boxed{\sqcup}$	0	1	$\sqcup$	$\cdots$
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$\triangleright$		

s	$\boxed{\triangleright}$	1	1	0	1	$\sqcup$	$\sqcup$	…
⋮				⋮		⋮		
s	$\triangleright$	1	1	0	1	$\boxed{\sqcup}$	$\sqcup$	…
q	$\triangleright$	1	1	0	$\boxed{\mathbf{1}}$	$\sqcup$	$\sqcup$	…
q_1	$\triangleright$	1	1	0	$\sqcup$	$\boxed{\sqcup}$	$\sqcup$	…
s	$\triangleright$	1	1	0	$\boxed{\sqcup}$	1	$\sqcup$	…
q	$\triangleright$	1	1	$\boxed{\mathbf{0}}$	$\sqcup$	1	$\sqcup$	…
q_0	$\triangleright$	1	1	$\sqcup$	$\boxed{\sqcup}$	1	$\sqcup$	…
s	$\triangleright$	1	1	$\boxed{\sqcup}$	0	1	$\sqcup$	…
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q_1	$\triangleright$	1	$\sqcup$	$\boxed{\sqcup}$	0	1	$\sqcup$	…
s	$\triangleright$	1	$\boxed{\sqcup}$	1	0	1	$\sqcup$	…
q	$\triangleright$	$\boxed{\mathbf{1}}$	$\sqcup$	1	0	1	$\sqcup$	…

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s	$\boxed{\triangleright}$	1	1	0	1	$\sqcup$	$\sqcup$	$\cdots$
$\vdots$				$\vdots$		$\vdots$		$\vdots$
s	$\triangleright$	1	1	0	1	$\boxed{\sqcup}$	$\sqcup$	$\cdots$
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q_0	$\triangleright$	1	1	$\sqcup$	$\boxed{\sqcup}$	1	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	$\boxed{\sqcup}$	0	1	$\sqcup$	$\cdots$
q	$\triangleright$	1	$\boxed{\mathbf{1}}$	$\sqcup$	0	1	$\sqcup$	$\cdots$
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s	$\triangleright$	1	$\boxed{\sqcup}$	1	0	1	$\sqcup$	$\cdots$
q	$\triangleright$	$\boxed{\mathbf{1}}$	$\sqcup$	1	0	1	$\sqcup$	$\cdots$
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q	$\triangleright$	1	1	$\boxed{\mathbf{0}}$	$\sqcup$	1	$\sqcup$	$\cdots$
q_0	$\triangleright$	1	1	$\sqcup$	$\boxed{\sqcup}$	1	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	$\boxed{\sqcup}$	0	1	$\sqcup$	$\cdots$
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s	$\triangleright$	1	$\boxed{\sqcup}$	1	0	1	$\sqcup$	$\cdots$
q	$\triangleright$	$\boxed{\mathbf{1}}$	$\sqcup$	1	0	1	$\sqcup$	$\cdots$
q_1	$\triangleright$	$\sqcup$	$\boxed{\sqcup}$	1	0	1	$\sqcup$	$\cdots$
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$\triangleright$		

s	$\boxed{\triangleright}$	1	1	0	1	$\sqcup$	$\sqcup$	$\cdots$
$\vdots$				$\vdots$		$\vdots$		$\vdots$
s	$\triangleright$	1	1	0	1	$\boxed{\sqcup}$	$\sqcup$	$\cdots$
q	$\triangleright$	1	1	0	$\boxed{\mathbf{1}}$	$\sqcup$	$\sqcup$	$\cdots$
q_1	$\triangleright$	1	1	0	$\sqcup$	$\boxed{\sqcup}$	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	0	$\boxed{\sqcup}$	1	$\sqcup$	$\cdots$
q	$\triangleright$	1	1	$\boxed{\mathbf{0}}$	$\sqcup$	1	$\sqcup$	$\cdots$
q_0	$\triangleright$	1	1	$\sqcup$	$\boxed{\sqcup}$	1	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	$\boxed{\sqcup}$	0	1	$\sqcup$	$\cdots$
q	$\triangleright$	1	$\boxed{\mathbf{1}}$	$\sqcup$	0	1	$\sqcup$	$\cdots$
q_1	$\triangleright$	1	$\sqcup$	$\boxed{\sqcup}$	0	1	$\sqcup$	$\cdots$
s	$\triangleright$	1	$\boxed{\sqcup}$	1	0	1	$\sqcup$	$\cdots$
q	$\triangleright$	$\boxed{\mathbf{1}}$	$\sqcup$	1	0	1	$\sqcup$	$\cdots$
q_1	$\triangleright$	$\sqcup$	$\boxed{\sqcup}$	1	0	1	$\sqcup$	$\cdots$
s	$\triangleright$	$\boxed{\sqcup}$	1	1	0	1	$\sqcup$	$\cdots$
q	$\boxed{\triangleright}$	$\sqcup$	1	1	0	1	$\sqcup$	$\cdots$

mvRt.tm		
	s	q
0	$s, 0, \rightarrow$	$q_0, \sqcup, \rightarrow$
1	$s, 1, \rightarrow$	$q_1, \sqcup, \rightarrow$
$\sqcup$	$q, \sqcup, \leftarrow$	
$\triangleright$	$s, \triangleright, \rightarrow$	$h, \triangleright, -$

	q_0	q_1
0		
1		
$\sqcup$	$s, 0, \leftarrow$	$s, 1, \leftarrow$
$\triangleright$		

s	$\boxed{\triangleright}$	1	1	0	1	$\sqcup$	$\sqcup$	$\cdots$
$\vdots$			$\vdots$		$\vdots$		$\vdots$	
s	$\triangleright$	1	1	0	1	$\boxed{\sqcup}$	$\sqcup$	$\cdots$
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Church's Thesis: The intuitive idea of **effectively computable** is **equivalent** to **Turing computable** and equivalently to computable by any of the above models.

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This is **better modeled** as a **TM** with a **bounded number of states**, and a **potentially infinite tape**.

Numbering Turing Machines

Turing machines can be encoded as **character strings** which can be encoded as **binary strings** which can be encoded as **natural numbers**.

TM_n	1	2	3	4
0	$1, 0, \rightarrow$	$3, \sqcup, \rightarrow$	$0, 0, -$	$0, 0, -$
1	$1, 1, \rightarrow$	$4, \sqcup, \rightarrow$	$0, 1, -$	$0, 1, -$
$\sqcup$	$2, \sqcup, \leftarrow$	$0, \sqcup, -$	$1, 0, \leftarrow$	$1, 1, \leftarrow$
$\triangleright$	$1, \triangleright, \rightarrow$	$0, \triangleright, -$	$0, \triangleright, -$	$0, \triangleright, -$

ASCII: $1, 0, \rightarrow; 1, 1, \rightarrow; 2, \sqcup, \leftarrow; 1, \triangleright, \rightarrow; ; \cdots\ 0, \triangleright, -$

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Countable listing of all TM's: $M_0, M_1, M_2, \cdots$

Universal Turing Machine

Theorem (Turing, 1936)

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All programs: $M_0, M_1, M_2, \ldots;\quad M_i(x)$ is the output of program i on input x.

The Halting Problem

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$$H(x, y) \quad \stackrel{\text{def}}{=} \quad \begin{cases} 1 & \text{if } M_x(y) \text{ eventually halts} \\ 0 & \text{otherwise} \end{cases}$$

Theorem (Turing, 1936)

The halting problem is not computable.

Proof.

Assume for the sake of a contradiction that $H(x, y)$ is computable and consider the following program:

$$D(x) \stackrel{\text{def}}{=} \textbf{if } H(x,x): \quad M_x(x) + 1 \quad \textbf{else}: \quad 0$$

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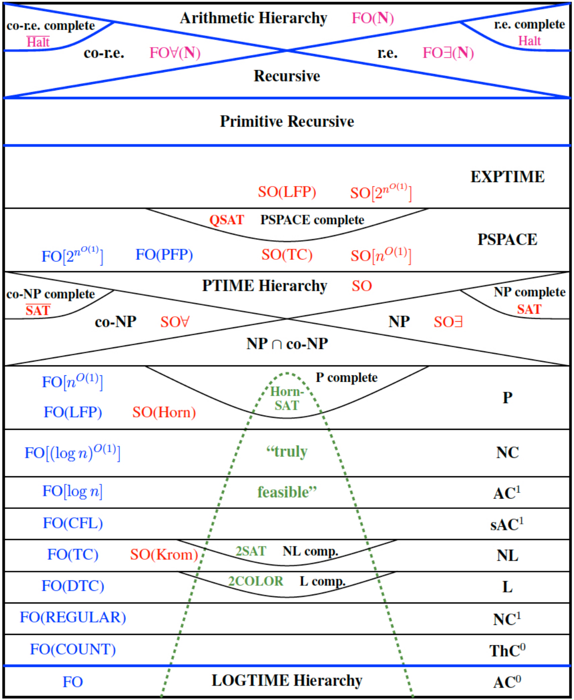
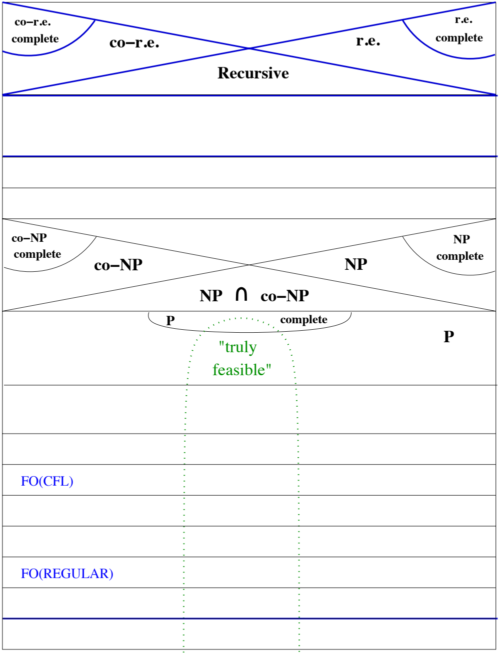
$\Rightarrow\Leftarrow$ $\square$



Arithmetic Hierarchy
FO(N)
co-r.e. complete
$\overline{\text{Halt}}$
r.e. complete
Halt
co-r.e.
FO∀(N)
r.e.
FO∃(N)
Recursive
Primitive Recursive
EXPTIME
SO(LFP)
$\text{SO}[2^{n^{O(1)}}]$
QSAT
PSPACE complete
PSPACE
$\text{FO}[2^{n^{O(1)}}]$
FO(PFP)
SO(TC)
$\text{SO}[n^{O(1)}]$
PTIME Hierarchy
SO
co-NP complete
$\overline{\text{SAT}}$
NP complete
SAT
co-NP
SO∀
NP
SO∃
NP ∩ co-NP
$\text{FO}[n^{O(1)}]$
P complete
Horn-SAT
P
FO(LFP)
SO(Horn)
$\text{FO}[(\log n)^{O(1)}]$
"truly
NC
$\text{FO}[\log n]$
feasible"
AC^1
FO(CFL)
sAC^1
FO(TC)
SO(Krom)
2SAT
NL comp.
NL
FO(DTC)
2COLOR
L comp.
L
FO(REGULAR)
NC^1
FO(COUNT)
ThC^0
FO
LOGTIME Hierarchy
AC^0

$$P = \bigcup_{k=1}^{\infty} \mathrm{DTIME}[n^k]$$

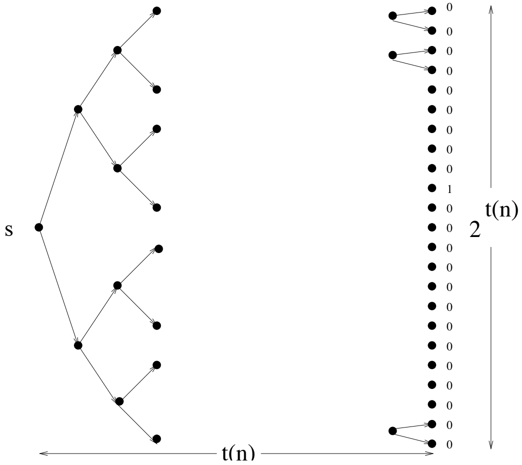


P is a good mathematical wrapper for “truly feasible”.

NTIME[$t(n)$]: a mathematical fiction

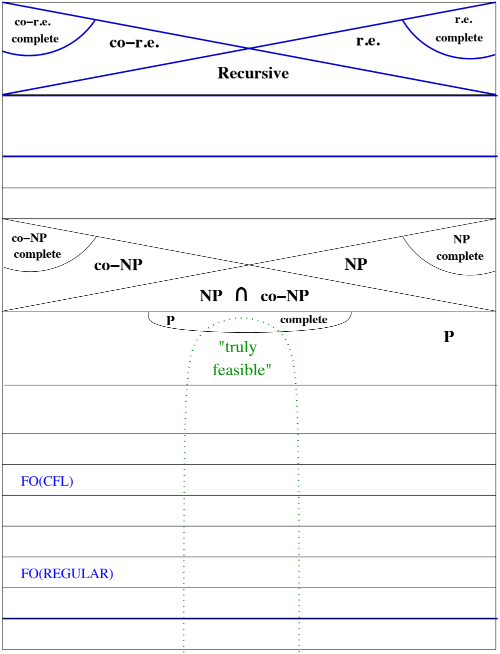
input w

$|w| = n$



$$\mathrm{NP} \quad = \quad \bigcup_{k=1}^{\infty} \mathrm{NTIME}[n^k]$$

Many optimization problems we want to solve are NP complete.



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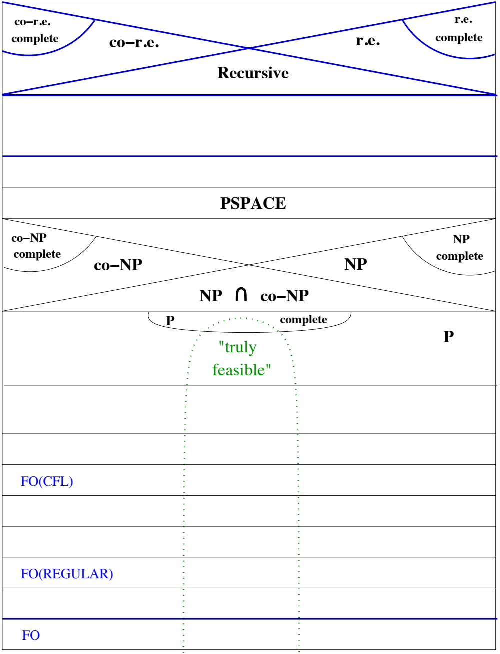

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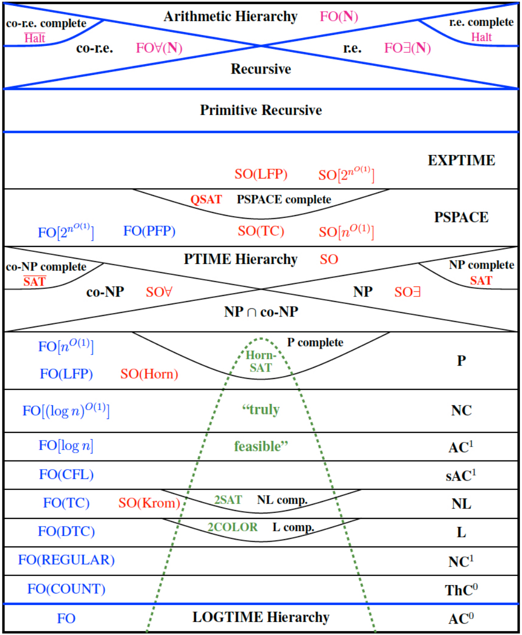

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CS311 Algorithms

CS513 Logic in CS

CS501 Theory of Computation