CS250: Discrete Math for Computer Science

L32: Nondeterministic Finite Automata: NFAs

Nondeterministic Finite Automata (NFA)

$$\left\{ w \in \{0,1\}^* \;\middle|\; w \text{ has } 001 \text{ or } 100 \right\} \;=\; \mathcal{L}((0|1)^*(001|100)(0|1)^*)$$

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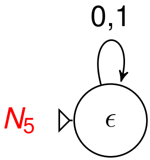

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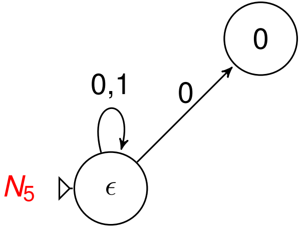

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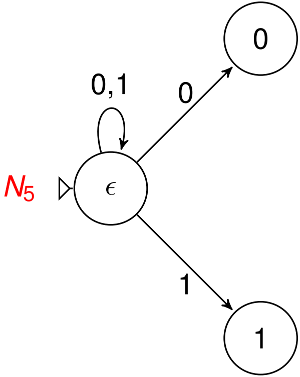

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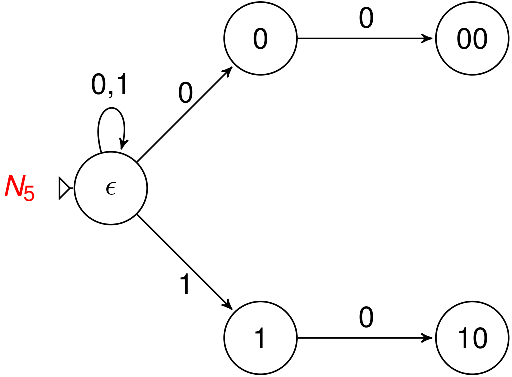

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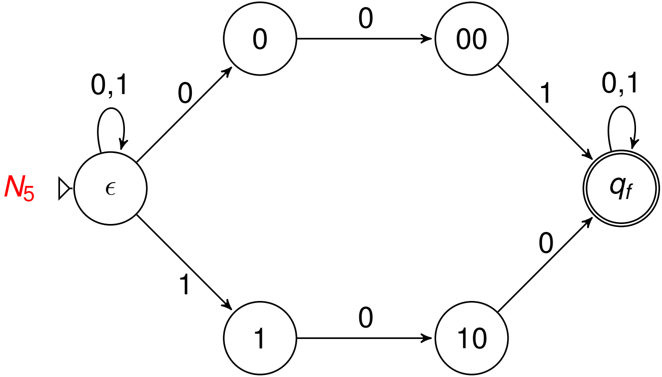

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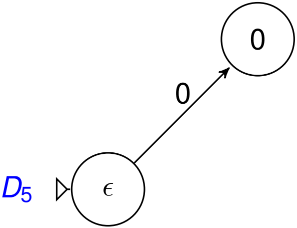

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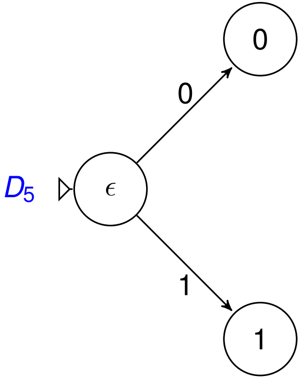

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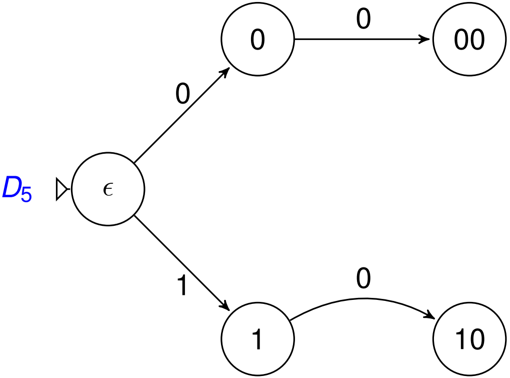

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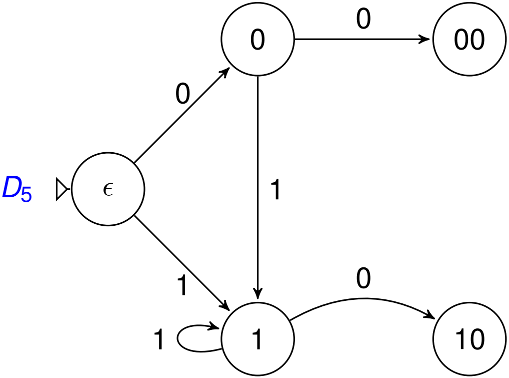

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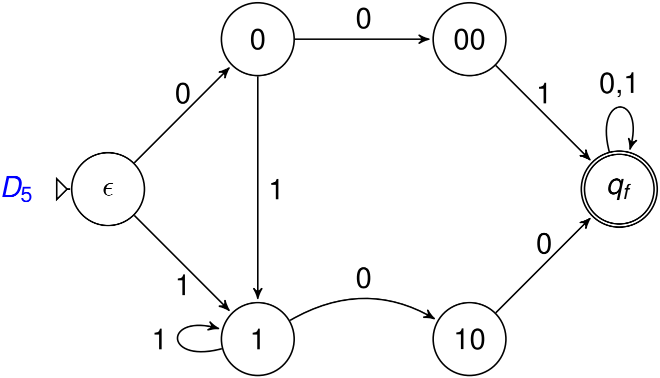

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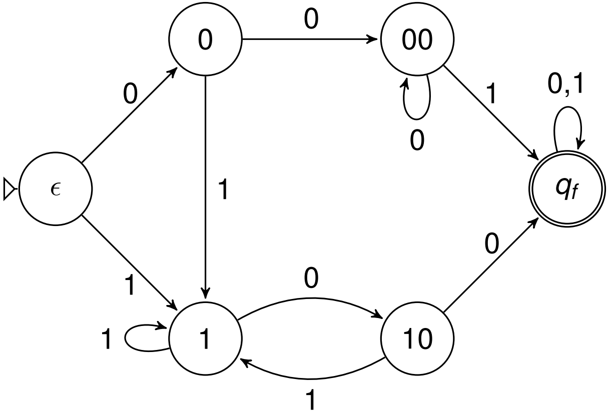
D_5



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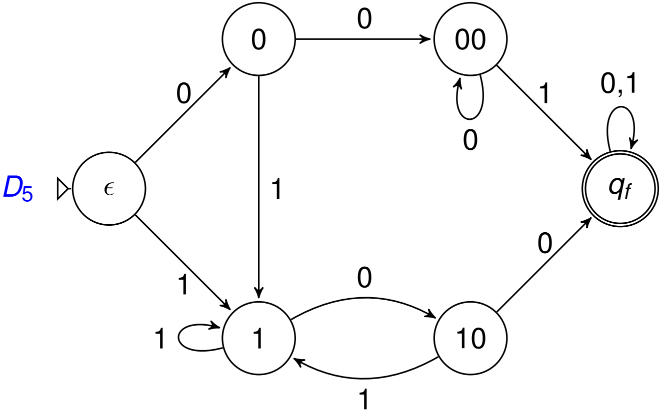
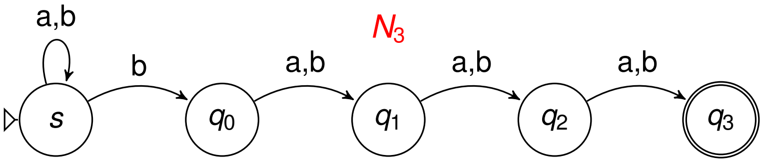


Every NFA has equivalent DFA; with **exponentially** more states.

NFA w.o. ϵ transitions



$$D = (Q, \Sigma, \delta, s, F) \qquad \delta : Q \times \Sigma \to Q$$

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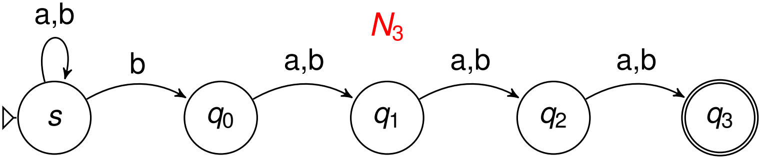

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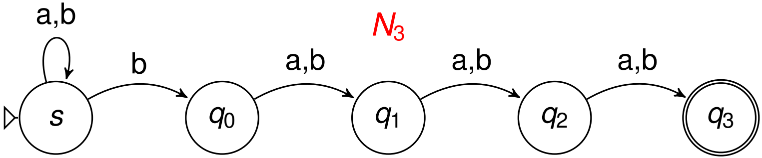

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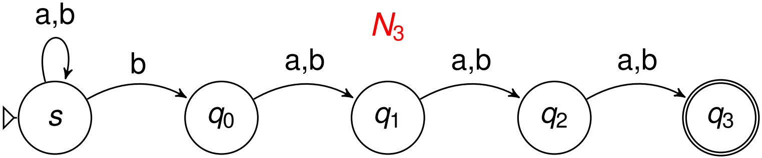

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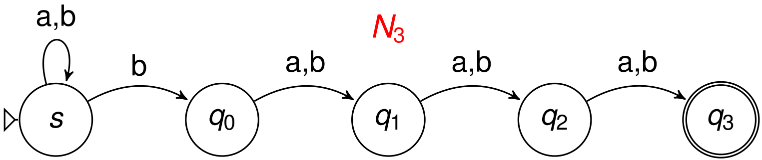

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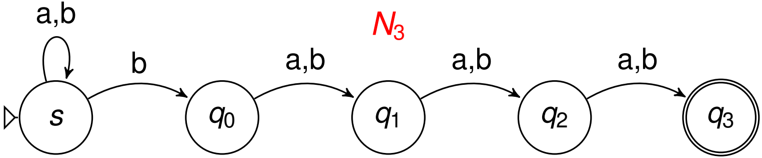

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power set of $Q = \wp(Q) \stackrel{\text{def}}{=} \{A \mid A \subseteq Q\}$

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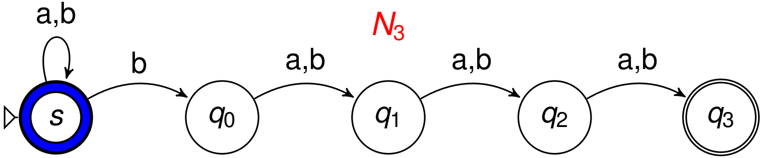

$\{s\}$

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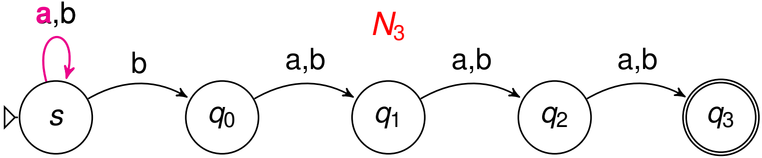

a

$\{s\}$

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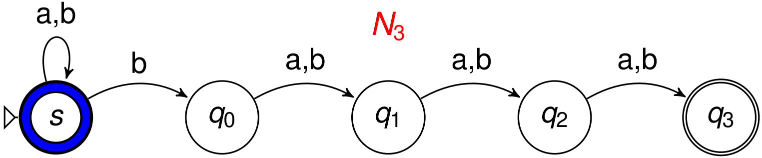

a

$\{s\}$ $\{s\}$

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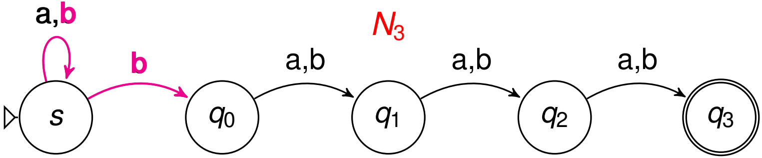

a	b
$\{s\}$	$\{s\}$

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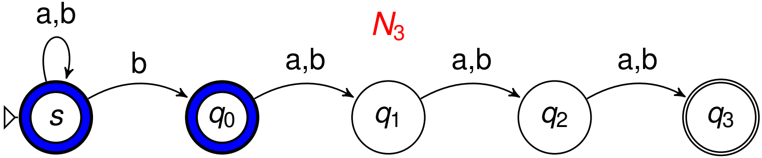

a **b**

$\{s\}$ $\{s\}$ $\{s, q_0\}$

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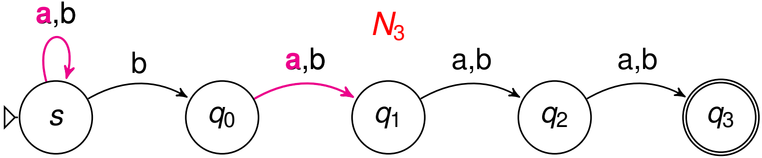

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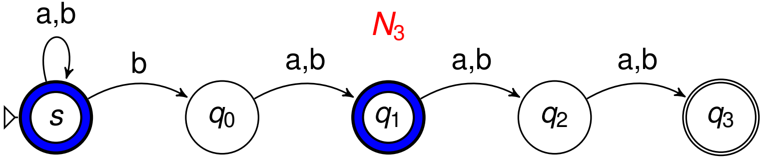

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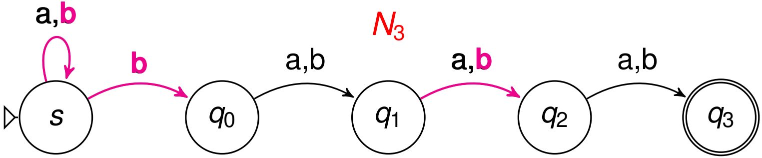

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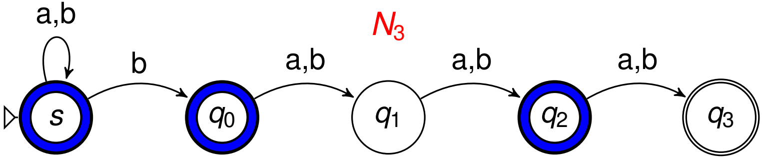

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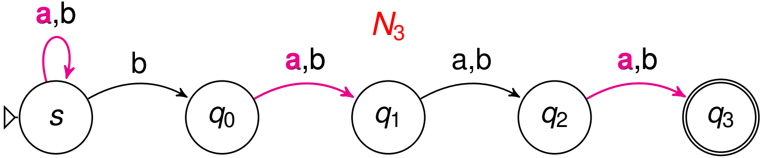

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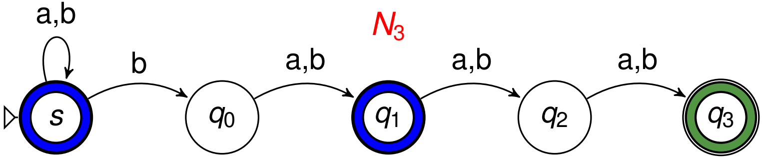

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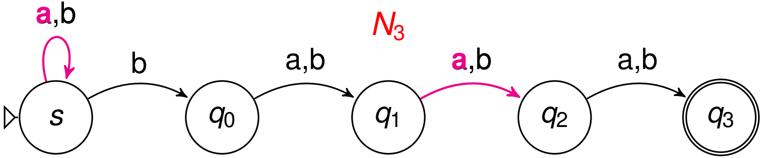

a b a b a a

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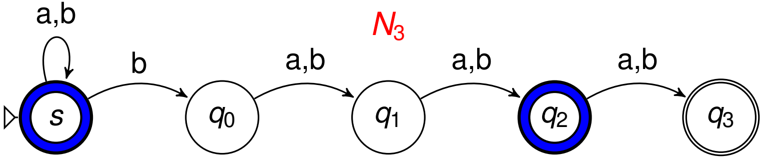

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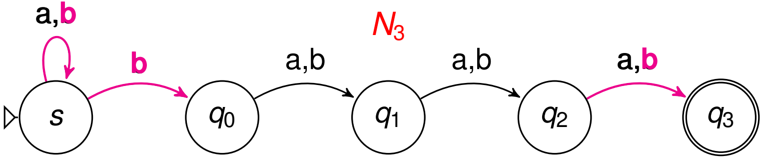

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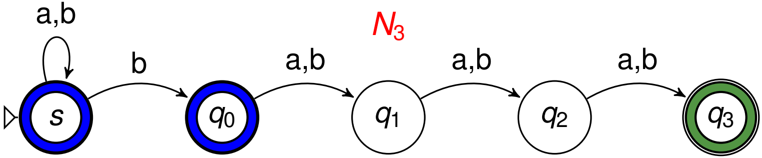

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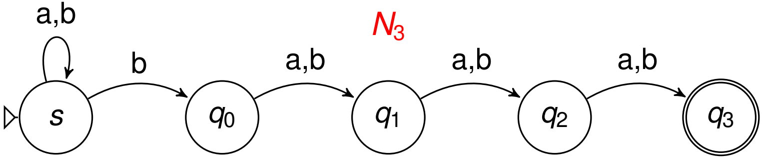

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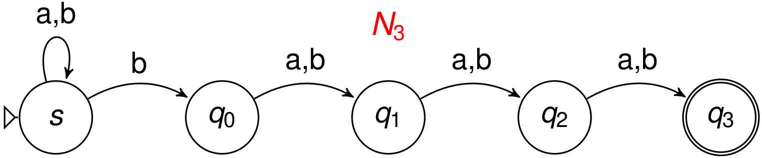

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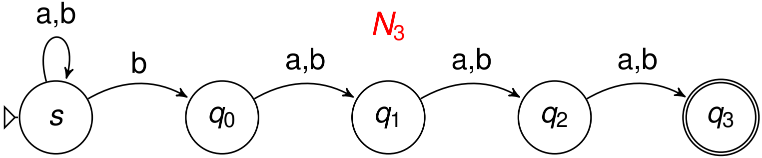

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iClicker 32.1 What is $\Delta_3(q_0, a)$?

A: q_1 **B:** $\{q_1\}$

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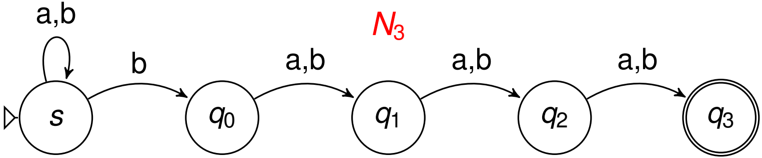

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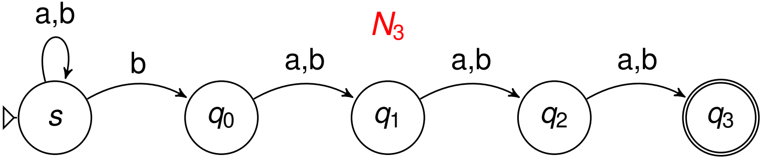

NFA may have 0 or more transitions marked *a* from any state.

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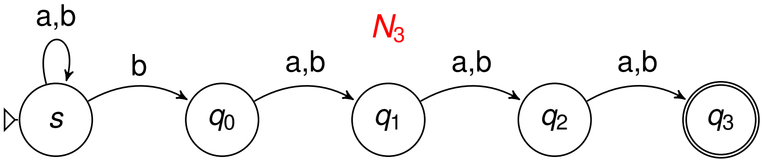

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Transitions may also be marked ϵ.

Nondeterministic Finite Automaton NFA

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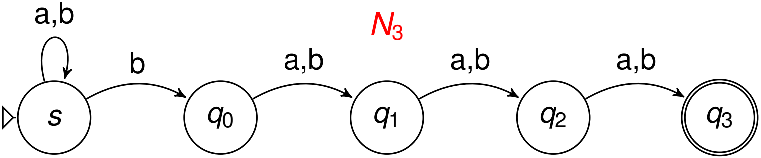
$$N_3 = (\{s, q_0, q_1, q_2, q_3\}, \{a, b\}, \Delta_3, s, \{q_3\})$$

$$\mathcal{L}(N_3) = \{a, b\}^\star \cdot \{b\} \cdot \{a, b\}^3$$



NFA may have 0 or more transitions marked *a* from any state.

Transitions may also be marked ϵ.

$$\mathcal{L}(N) = \left\{ w \in \Sigma^\star \;\middle|\; \Delta^\star(s, w) \cap F \neq \emptyset \right\}$$

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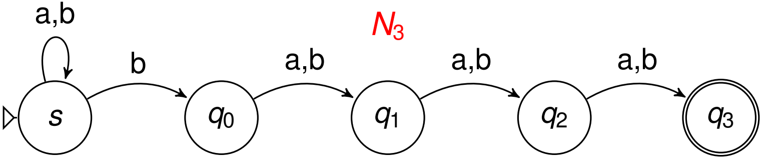

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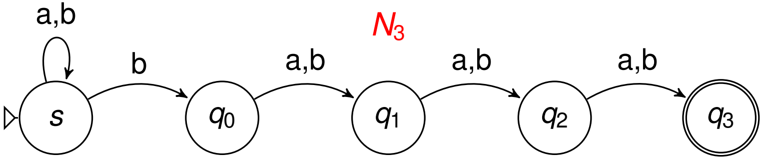
$$\mathcal{L}(N) = \left\{ w \in \Sigma^\star \;\middle|\; \Delta^\star(s, w) \cap F \neq \emptyset \right\}$$

$w \in \mathcal{L}(N)$ iff N **can** go from s to F while reading w.

NFAs can be much smaller and simpler



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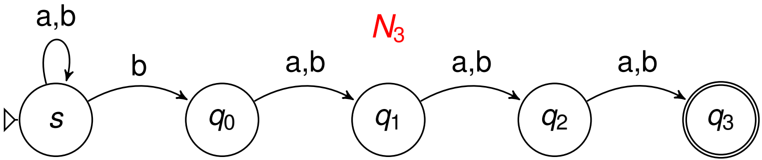

How many states in **smallest DFA** D_3 s.t. $\mathcal{L}(D_3) = \mathcal{L}(N_3)$?

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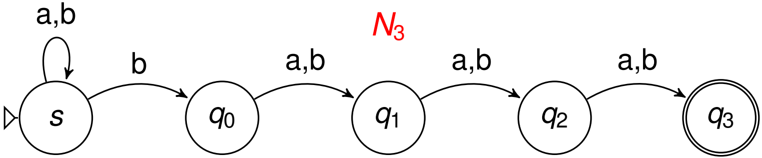

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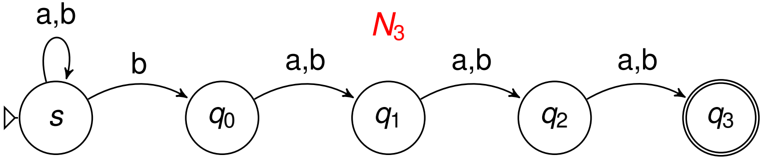

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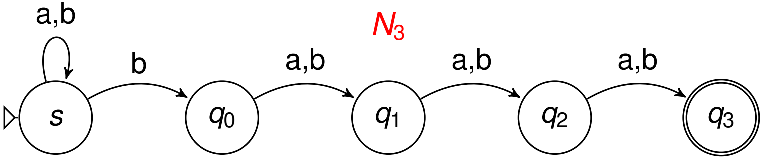

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\begin{aligned}
D_3 &= (\{a,b\}^4, \{a,b\}, \delta_3, aaaa, \{b\}\{a,b\}^3) \\
\delta_3(s_1 s_2 s_3 s_4, c) &= s_2 s_3 s_4 c
\end{aligned}
$$

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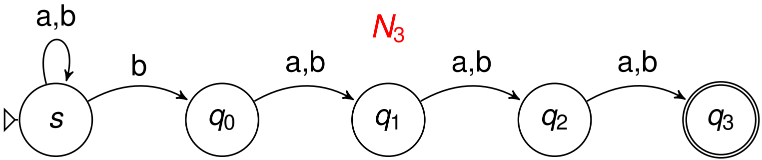

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N_3 has 5 states; D_3 requires 2^4 states.

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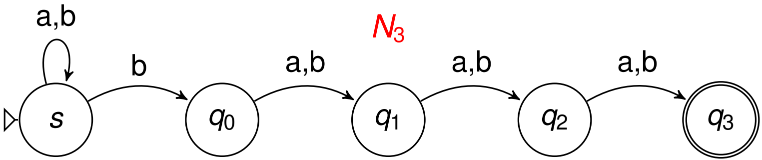

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Generalize N_3 to N_r; $\mathcal{L}(N_r) = \{a, b\}^\star \cdot \{b\} \cdot \{a, b\}^r$

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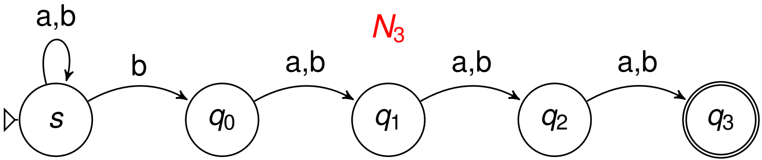

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N_r has $r+2$ states; D_r requires 2^{r+1} states.

DFA needs exponentially more states than equiv NFA

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Suppose: $\mathcal{L}(D) = \mathcal{L}(N_3) = \{a, b\}^\star \cdot \{b\} \cdot \{a, b\}^3$

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proof: assume not

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Consider D's state after starting in s and reading w_i

$\delta^\star(s, w_1) \quad \delta^\star(s, w_2) \quad \cdots \quad \delta^\star(s, w_{16})$

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Let $w = a^{i-1}, \quad xw \notin \mathcal{L}(D), \quad yw \in \mathcal{L}(D)$

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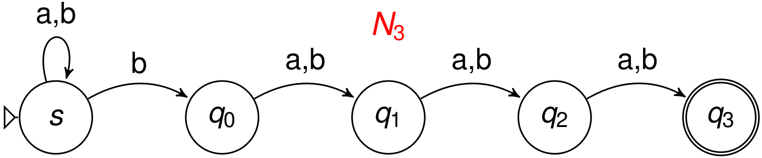
Same state is in F and not in F. $\square$

Thm. DFAs **sometimes require exponentially more states** than equivalent NFAs.

D_3 **must remember last 4 symbols**, requiring 2^4 states.

Same argument shows D_r requires 2^r states.

N_3



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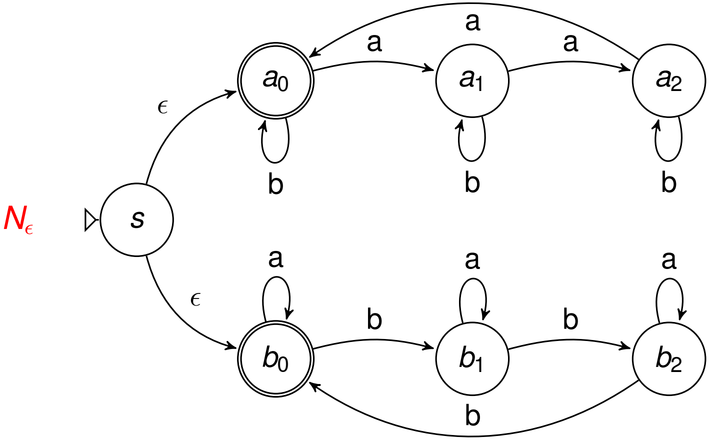
NFAs with ϵ-moves



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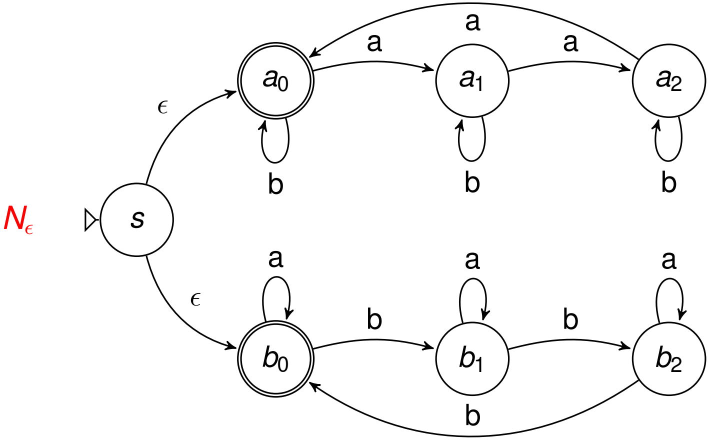
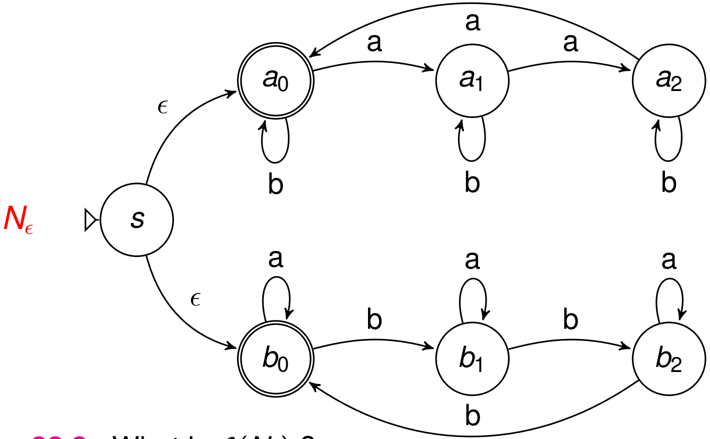


iClicker 32.2 Is $\epsilon \in \mathcal{L}(N_\epsilon)$?

A: yes **B: no**

NFAs with ϵ-moves

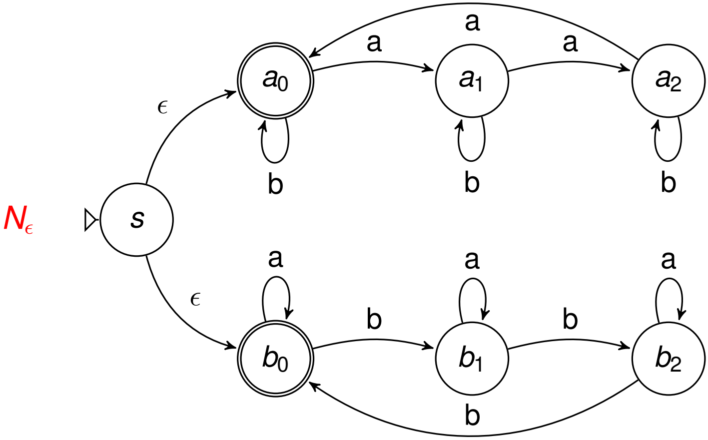


iClicker 32.3 What is $\mathcal{L}(N_\epsilon)$?

A: $\left\{ w \in \{a, b\}^\star \;\middle|\; \#_a(w) \equiv 0 \pmod 3 \;\wedge\; \#_b(w) \equiv 0 \pmod 3 \right\}$
B: $\left\{ w \in \{a, b\}^\star \;\middle|\; \#_a(w) \equiv 0 \pmod 3 \;\vee\; \#_b(w) \equiv 0 \pmod 3 \right\}$

Eliminating ϵ-moves

Thm: For every n-state NFA N there is an n-state NFA N' that has no ϵ moves and $\mathcal{L}(N) = \mathcal{L}(N')$.



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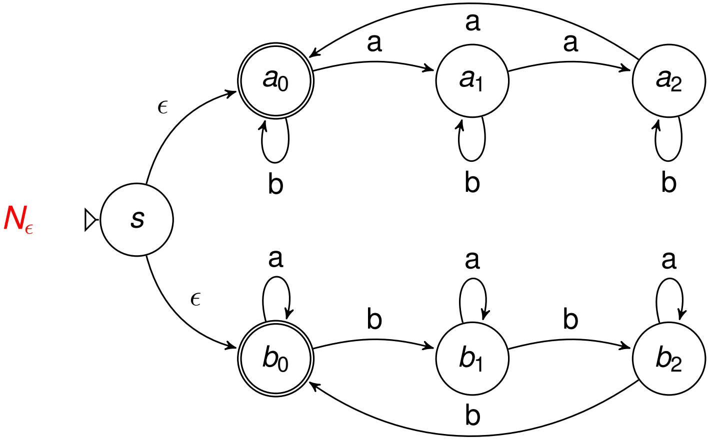
$$\Delta'(q, a) = \left\{ q' \;\middle|\; q \overset{\epsilon^\star a \epsilon^*}{\longrightarrow} q' \right\} \quad F' = \left\{ q \;\middle|\; \exists f \in F(q \overset{\epsilon^\star}{\longrightarrow} f) \right\}$$



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