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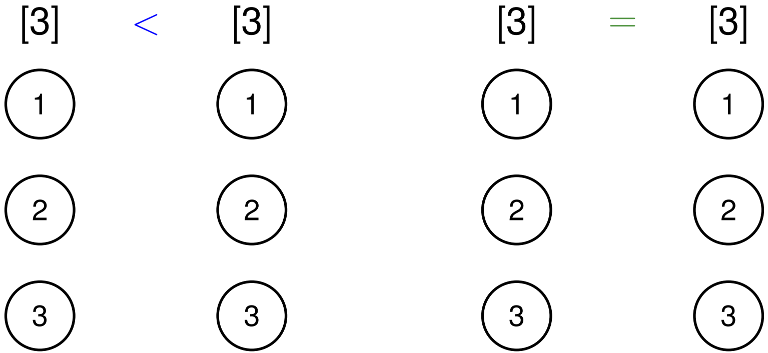
- **We'll talk about this later.**

Arrow Diagram of a Relation

For relation R from A to B, draw an arrow from a to b iff aRb.



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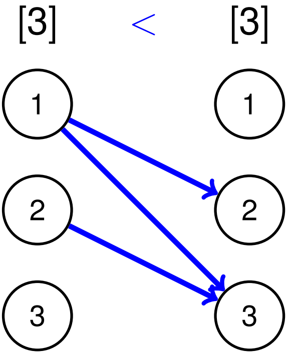

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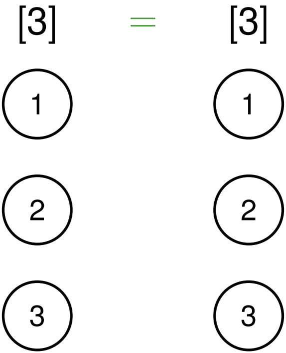

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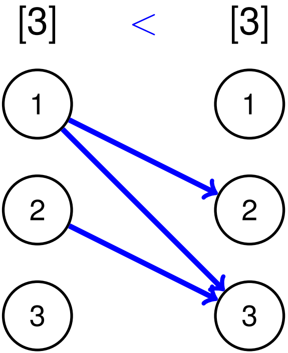

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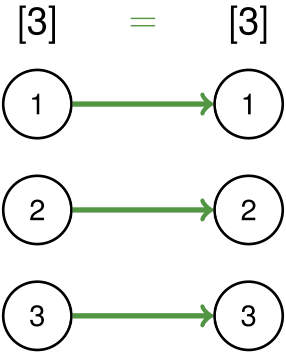

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The Divides Relation: "|"

$$a|b \;\; (a \text{ **divides** } b) \quad \text{iff} \quad \exists d \in \mathbf{Z}\,(a \cdot d = b) \quad \text{iff} \quad \frac{b}{a} \in \mathbf{Z}$$

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$3 \nmid 10$ because $\nexists d \in \mathbf{Z}\,(3 \cdot d = 10)$ $\qquad \dfrac{10}{3} \notin \mathbf{Z}$

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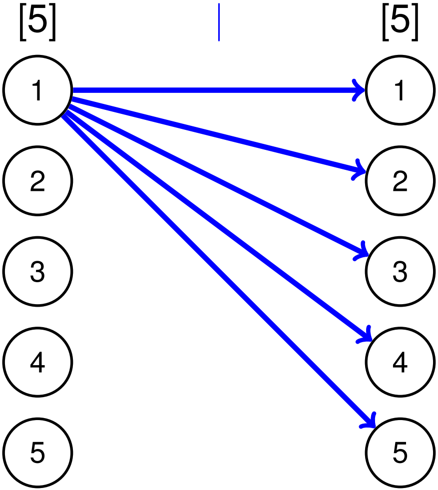
iClicker 3.1 True or False: $\forall z \in \mathbf{Z}\; 1|z$?

A: True
B: False

Arrow Diagram of Divides Relation



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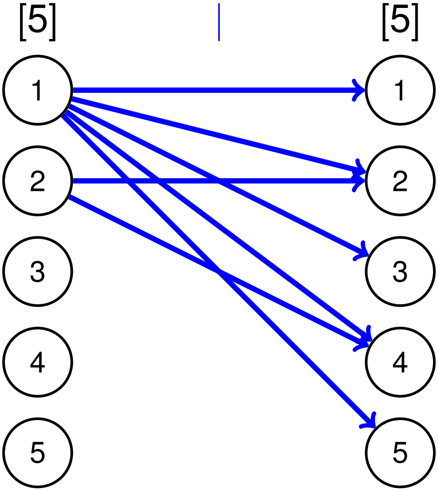

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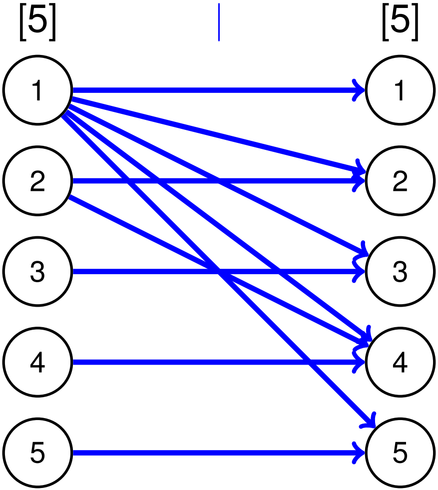

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Functions $f : A \to B$ $\quad (a, b) \in f$ iff $f(a) = b$

Def: f is a **function** from A to B iff $f \subseteq A \times B$, and

f is **defined** on domain A: $\forall a \in A\, \exists b \in B\, (a, b) \in f$, and

f is **single valued**: $\forall (a, b), (a', b') \in f\ (a = a' \ \to\ b = b')$.

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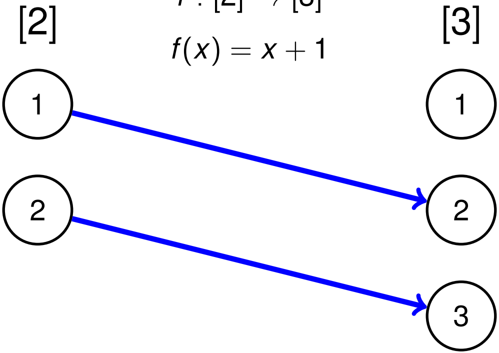
$f : [2] \to [3]$

$f(x) = x + 1$



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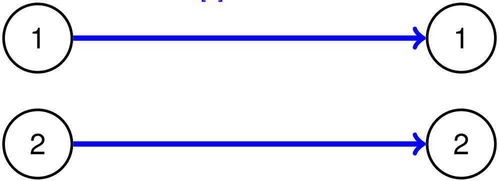
$[2]$ $\qquad \mathrm{id}_{[2]} : [2] \to [2]$ $\qquad [2]$

$\mathrm{id}_{[2]}(x) = x$



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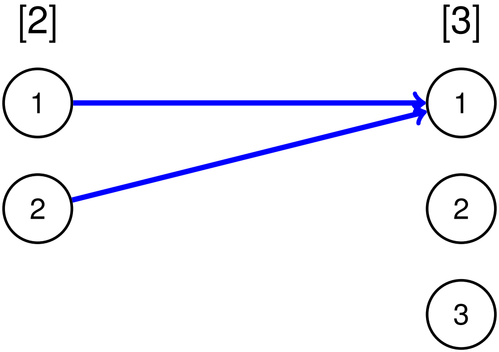
iClicker 3.2 Let $g = \{(1, 1), (2, 1)\}$. Is $g : [2] \to [3]$?

A: Yes **B: No**



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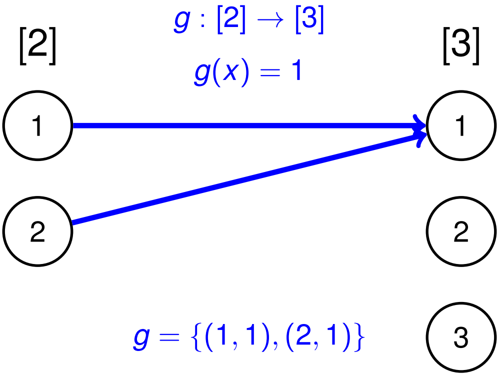

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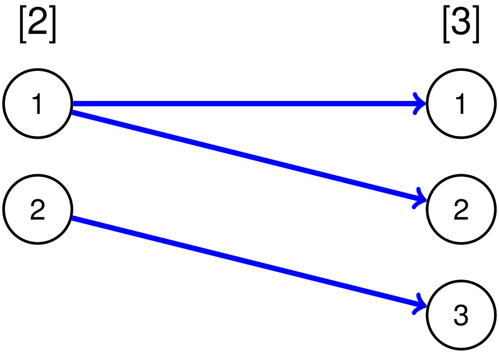

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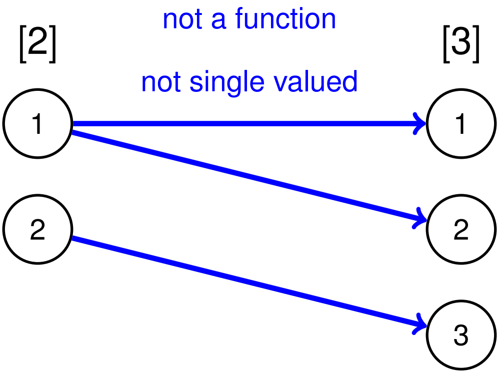
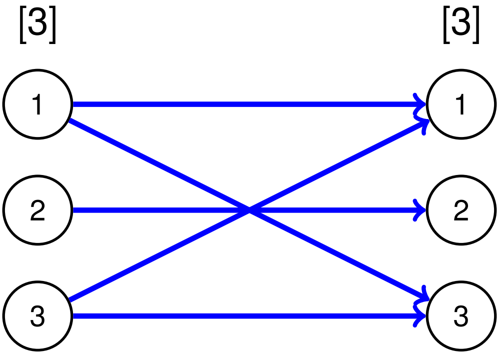
A: Yes **B: No**



R3 Quiz Answers

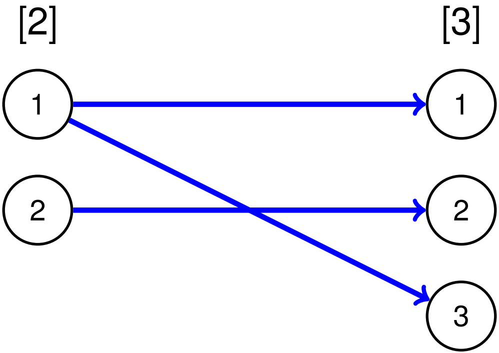
1. $D2D_3$ from [3] to [3] $\quad |\mathbf{D2D_3}| = 5$
2. $D2D_3 : [3] \to [3]$? **False: not single valued**



R3 Quiz Answers

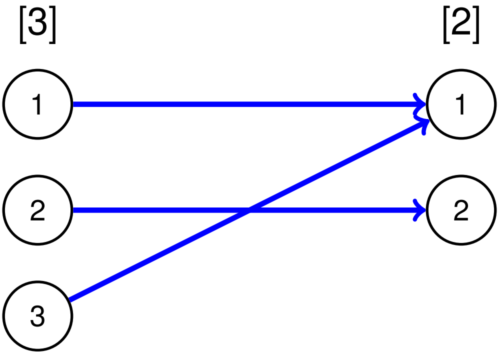
3. $\mathrm{D2D}_{2,3}$ from [2] to [3] $\quad |\mathbf{D2D}_{2,3}| = 3$

4. $\mathrm{D2D}_{2,3} : [2] \to [3]$? $\quad$ **False: not single valued**



R3 Quiz Answers

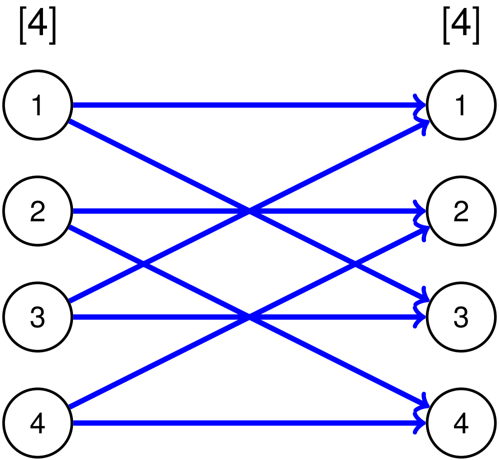
5. $D2D_{3,2}$ from [3] to [2] $\quad |\mathbf{D2D}_{3,2}| = 3$
6. $D2D_{3,2} : [3] \to [2]$? **True**



R3 Quiz Answers

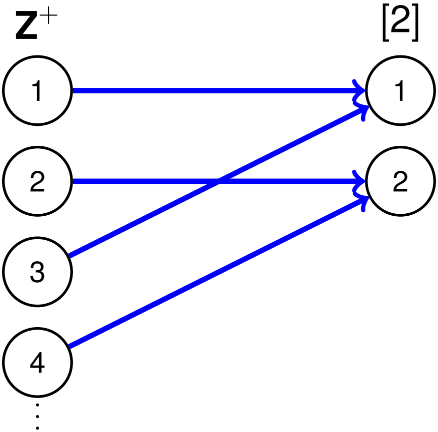
7. D2D_4 from [4] to [4] $\quad |\textbf{D2D}_4| = 8$

8. $\text{D2D}_4 : [4] \to [4]$? **False: not single valued**



R3 Quiz Answers

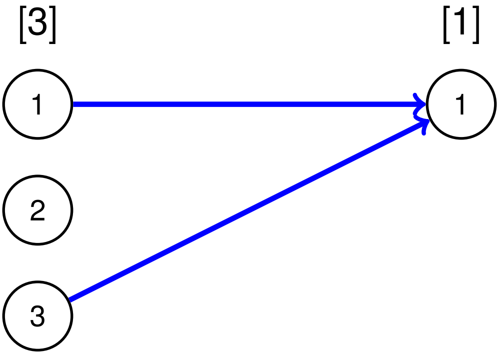
9. $\text{D2D}_{\infty,2}$ from $\mathbf{Z}^+$ to [2] $|\textbf{D2D}_{\infty,2}| = \aleph_0$ **(infinite)**
10. $\text{D2D}_{\infty,2} : \mathbf{Z}+ \to [2]$? **True**



R3 Quiz Answers

11. $\mathrm{D2D}_{3,}$ from [3] to [1] $\quad |\mathbf{D2D}_{3,}| = 2$

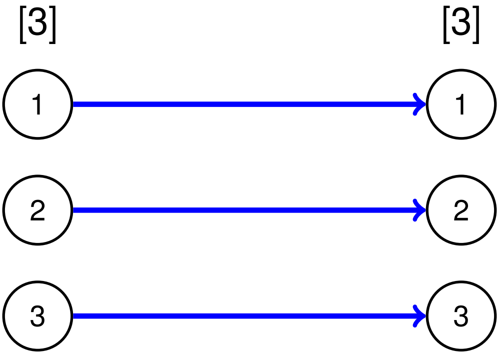
12. $\mathrm{D2D}_{3,1} : [3] \to [1]$? **False: not defined on** $2 \in [3]$



R3 Quiz Answers

13. $D3D_3$ from [3] to [3] $\quad |\mathbf{D3D}_3| = 3$

14. $D3D_3 : [3] \to [3]$? **True**



R3 Quiz Answers

$$R_8 = \left\{(a,b) \in (\mathbf{R}^+)^2 \;\middle|\; b^2 = a\right\}$$

15. $R_8 \subseteq \mathbf{R}^+ \times \mathbf{R}^+$ $\quad |R_8| \;=\; |\mathbf{R}|$ **(infinite)**

16. $R_8 : \mathbf{R} \to \mathbf{R}$? $\quad$ **True:** $R_8(a) = \sqrt{a}$

$$R_9 = \left\{(a,b) \in (\mathbf{R})^2 \;\middle|\; b^2 = a\right\}$$

17. $R_9 \subseteq \mathbf{R} \times \mathbf{R}$ $\quad |R_9| \;=\; |\mathbf{R}|$ **(infinite)**

18. $R_9 : \mathbf{R} \to \mathbf{R}$? $\quad$ **False:** $R_9(a)$ **is undefined when** $a < 0$

$$R_{10} = \left\{(a,b) \in (\mathbf{R})^2 \;\middle|\; a^2 = b\right\}$$

19. $R_{10} \subseteq \mathbf{R} \times \mathbf{R}$ $\quad |R_{10}| \;=\; |\mathbf{R}|$ **(infinite)**

20. $R_{10} : \mathbf{R} \to \mathbf{R}$? $\quad$ **True:** $R_{10}(a) = a^2$