CS250: Discrete Math for Computer Science

L27: Cryptography and RSA

Recall: Fermat's Little Theorem

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$$1 \equiv a^{p-1} \pmod p \qquad \square$$

Euler's phi function, $\varphi(n) = |\mathbf{Z}_n^*|$

n	$\varphi(\mathbf{n})$	n	$\varphi(\mathbf{n})$	n	$\varphi(\mathbf{n})$
2	**1**	11	**10**	20	**8**
3	**2**	12	**4**	21	**12**
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If gcd$(a, b) = 1$**,**

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For primes, $p \neq q$,

$$\varphi(pq) = (p-1)(q-1)$$

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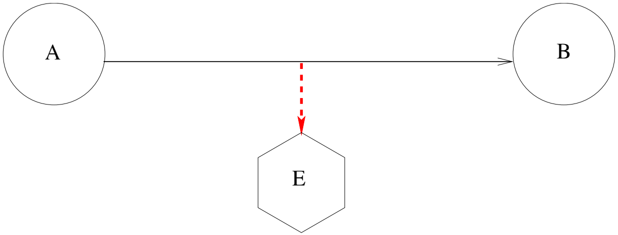
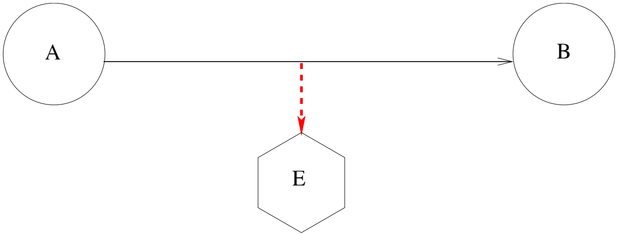
Cryptography

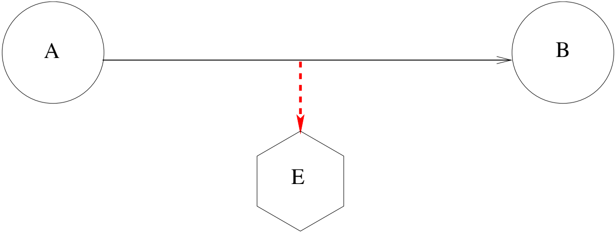


Cryptography



One-Time Pad: a perfectly secure cryptosystem

Cryptography



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$p \in \{0,1\}^n$ $\quad m \in \{0,1\}^n \quad = \quad$ binary strings of length n

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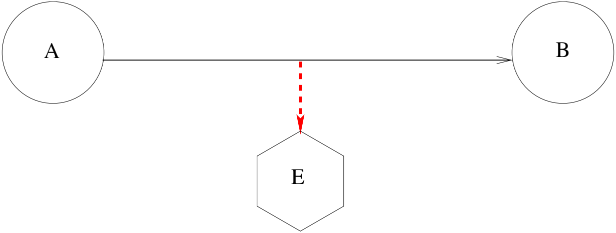

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$$E(p,x) = p \oplus x$$

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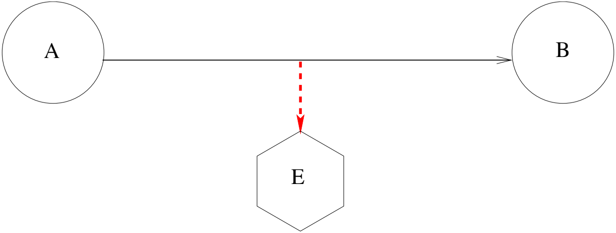

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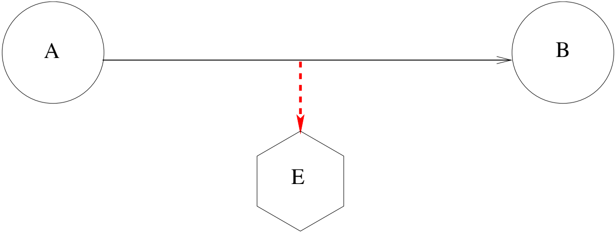

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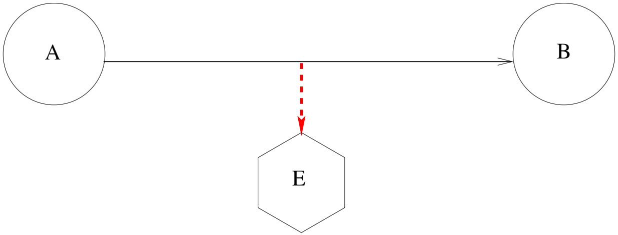
$$
\begin{aligned}
E(p,x) &= p \oplus x \\
D(p,x) &= p \oplus x \\
D(p,E(p,m)) &= p \oplus (p \oplus m) \quad = \quad m
\end{aligned}
$$

Cryptography



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$$D(p, E(p,m)) = p \oplus (p \oplus m) = m$$

Encryption and **decryption** functions are the same:
bitwise **exclusive or** with **random**, **secret** one-time pad, p.

One-Time Pad, Continued

p	0	1	1	0	0	1	0	1	0	1

$$E(p, m) \ = \ p \oplus m \qquad\qquad D(p, s) \ = \ p \oplus s$$

One-Time Pad, Continued

p	0	1	1	0	0	1	0	1	0	1
m	0	0	0	0	1	1	1	1	0	0

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One-Time Pad, Continued

p	0	1	1	0	0	1	0	1	0	1
m	0	0	0	0	1	1	1	1	0	0
$E(p, m)$	0	1	1	0	1	0	1	0	0	1

$$E(p, m) = p \oplus m \qquad D(p, s) = p \oplus s$$

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Thm: If p is **chosen at random** and **known only** by A and B,

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Then $E(p,m)$ provides **no information** about m

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except perhaps its length.

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Do not use p more than once!

Public-Key Cryptography

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RSA [Rivest, Shamir, Adleman, 1976]

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without, **we believe**, divulging any **useful** information about your credit card number, or what you bought.

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Public Key Cryptography is a **theoretical underpinning** for possible computer security even over the web.