CS250: Discrete Math for Computer Science

L25: Binary Relations and Digraphs

1:1 Functions

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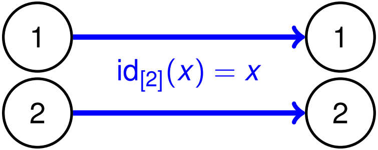

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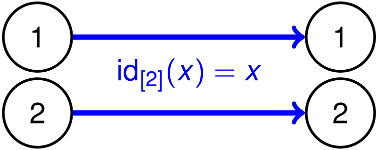

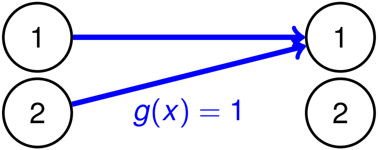

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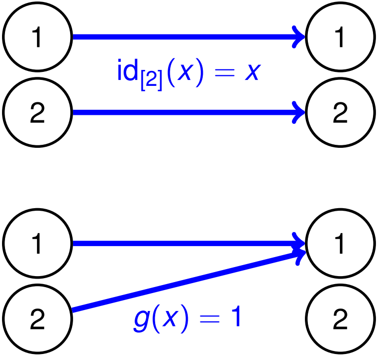

iClicker 25.1 Which functions on the left are 1:1 ?

A: just $\mathrm{id}_{[2]}$

B: just g

C: both of them

D: neither of them

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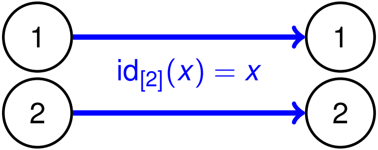

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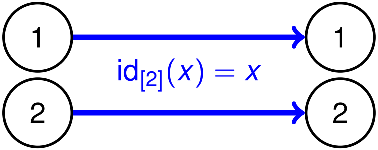

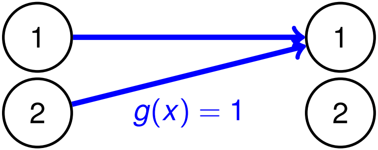

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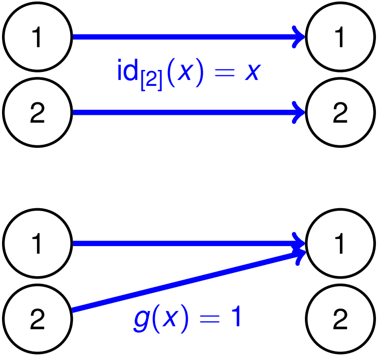

iClicker 25.2 Which functions on the left are onto ?

A: just $\mathrm{id}_{[2]}$

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Domain, Range, and Co-Domain

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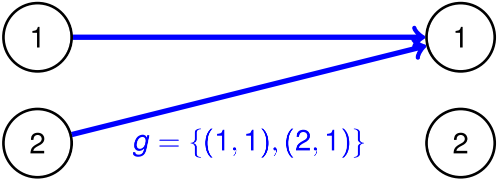

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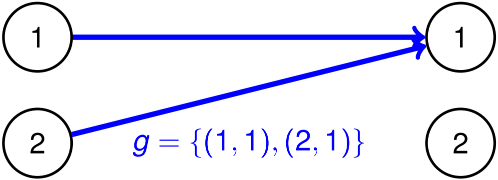

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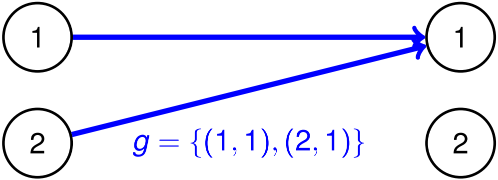

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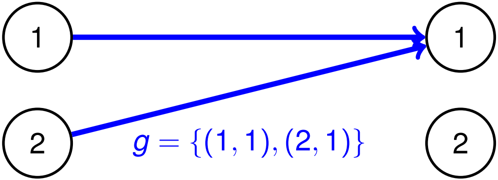

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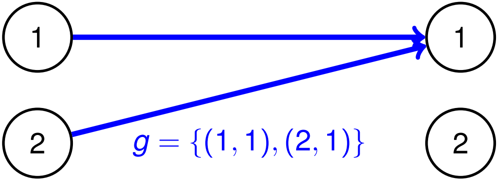

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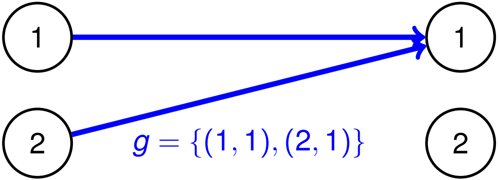

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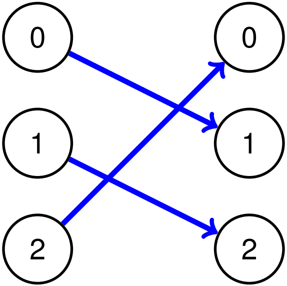

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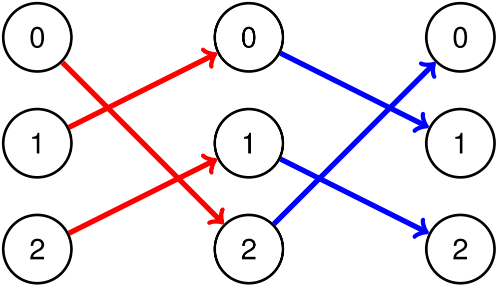

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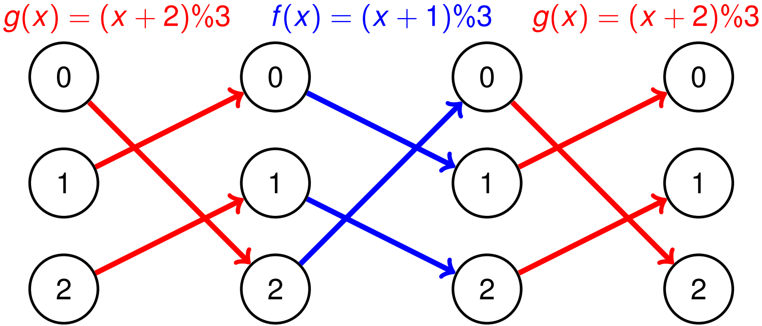

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Thm. $f : A \to B$ has an inverse iff f is 1:1 and onto.

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Does $f_2 : \mathbf{N} \to \mathbf{N}, \quad f_2(n) = n + 1$ have an inverse?

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Thm. $f : A \to B$ has an inverse iff f is 1:1 and onto.

Proof: Already argued it is necessary that f is 1:1 and onto.

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Let $f^T \stackrel{\mathrm{def}}{=} \left\{(b, a) \;\middle|\; (a, b) \in f\right\}$ **transpose** of f.

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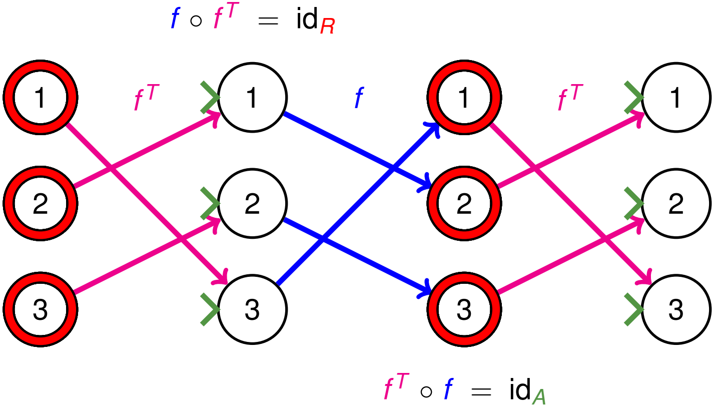
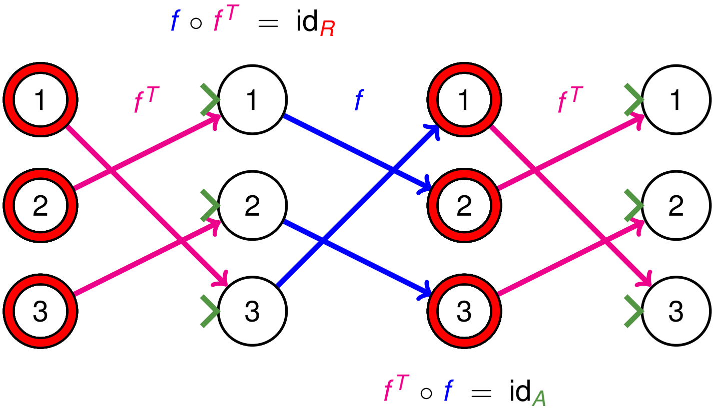
$f^T : B \to A$ and $f^T \circ f = \mathrm{id}_A$ and $f \circ f^T = \mathrm{id}_B$ $\square$

Claim: $f \circ f^T = \mathrm{id}_R$ and $f^T \circ f = \mathrm{id}_A$



Claim: $f \circ f^T = \mathrm{id}_R$ and $f^T \circ f = \mathrm{id}_A$

Proof: f is onto: $\forall y \in R \; \exists x \in A \; (y, x) \in f^T$, thus $f \circ f^T(y) = y$



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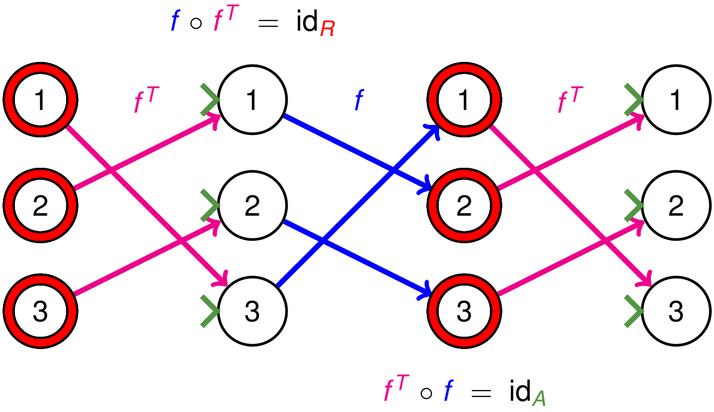
f is 1:1: $\forall x \in A \quad f^T \circ f(x) = x$ $\square$



Binary Relations from V to V

Def. A **directed graph** (**digraph**), $G = (V^G, E^G)$ is a world of vocabulary $\Sigma_{\mathrm{g}} = (E^2;)$. Thus a digraph, G, is just a binary relation, E^G, from V^G to V^G.

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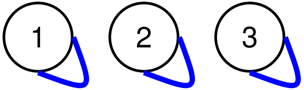
$=^{[3]}$



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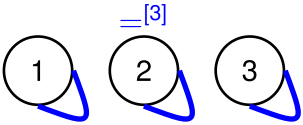

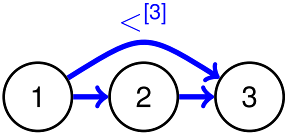

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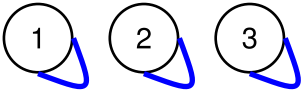
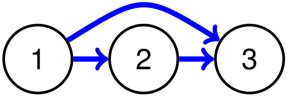
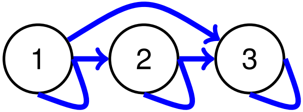


$<^{[3]}$



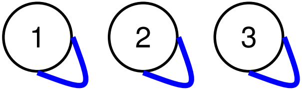


$\leq^{[3]}$

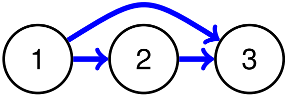
Binary Relations from V to V

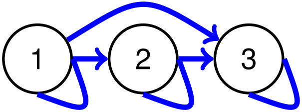
Def. A **directed graph** (**digraph**), $G = (V^G, E^G)$ is a world of vocabulary $\Sigma_g = (E^2;)$. Thus a digraph, G, is just a binary relation, E^G, from V^G to V^G.

$=^{[3]}$

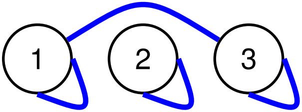


$<^{[3]}$



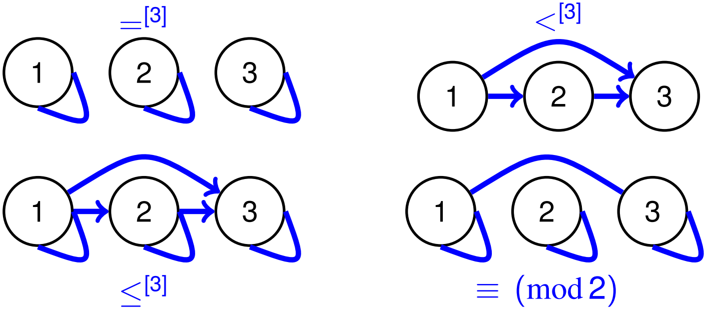


$\leq^{[3]}$



$\equiv \pmod 2$

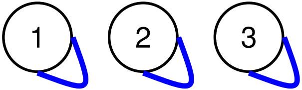
reflexive $\equiv \forall x\, E(x,x)$



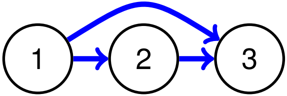
reflexive $\equiv \forall x\, E(x,x)$

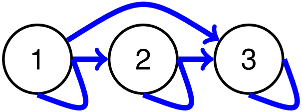
symmetric $\equiv \forall xy\, (E(x,y) \rightarrow E(y,x))$

$=^{[3]}$

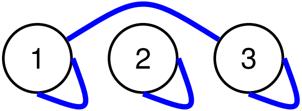


$<^{[3]}$





$\leq^{[3]}$



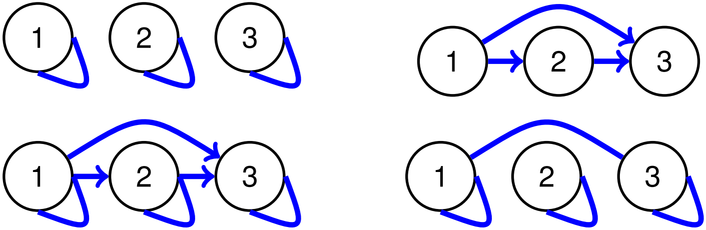
$\equiv \pmod 2$

$$\textbf{reflexive} \equiv \forall x\, E(x,x)$$

$$\textbf{symmetric} \equiv \forall xy\,(E(x,y) \to E(y,x))$$

$$\textbf{transitive} \equiv \forall xyz\,(E(x,y) \wedge E(y,z) \to E(x,z))$$

$=^{[3]}$ $\quad$ $<^{[3]}$



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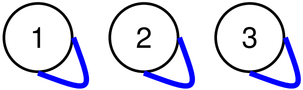
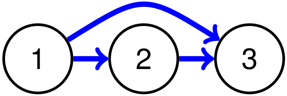
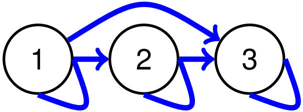
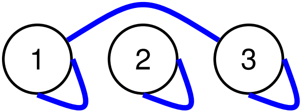


$<^{[3]}$





$\leq^{[3]}$



$\equiv \pmod 2$

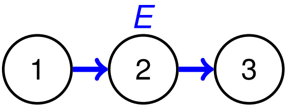
iClicker 25.3 Which are Reflexive, Symmetric and Transitive ?

A: all **B: just $\equiv \pmod 2$** **C: $=^{[3]}$ and $\equiv \pmod 2$**

D: all but $<^{[3]}$

Transitive Closure

Def. Transitive Closure E^+ is the smallest **transitive** relation containing E.



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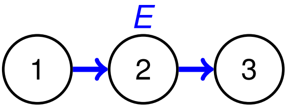
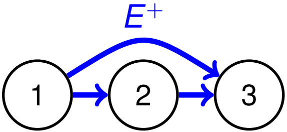
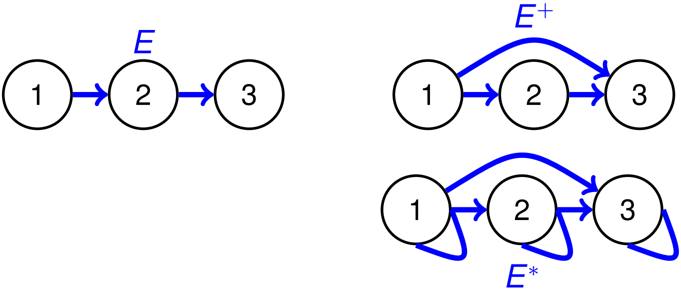




Transitive Closure

Def. Transitive Closure E^+ is the smallest **transitive** relation containing E. The **Reflexive Transitive Closure** E^* is the smallest **reflexive** and **transitive** relation containing E.



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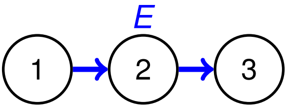

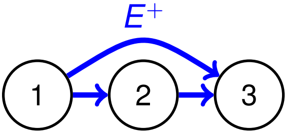

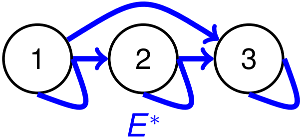

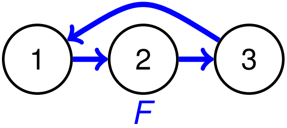

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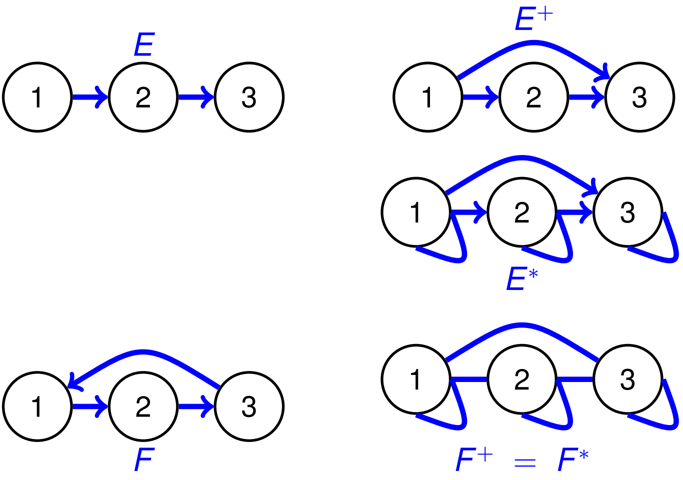
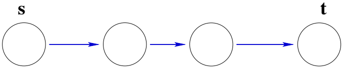


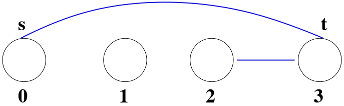
Connectivity

$$\textbf{conn} \quad \equiv \quad \forall xy \, E^{\star}(x, y)$$

Undirected graph G is **connected** iff $\mathcal{G} \models$ **conn**.

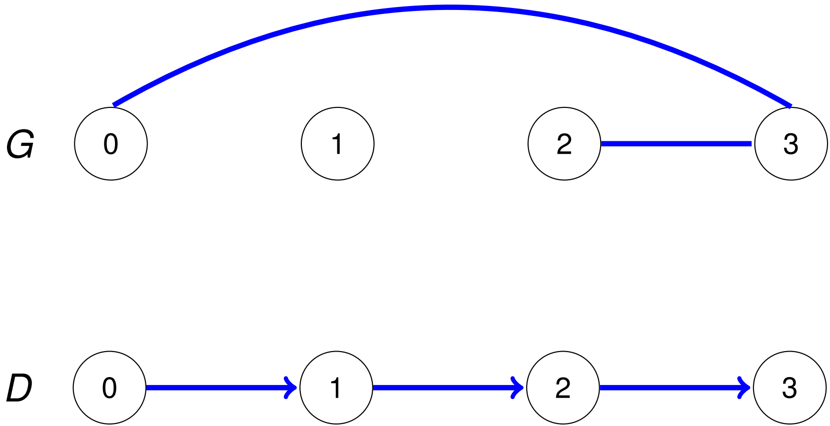
Directed graph G is **strongly connected** iff $\mathcal{G} \models$ **conn**.

G_1 

G_4 

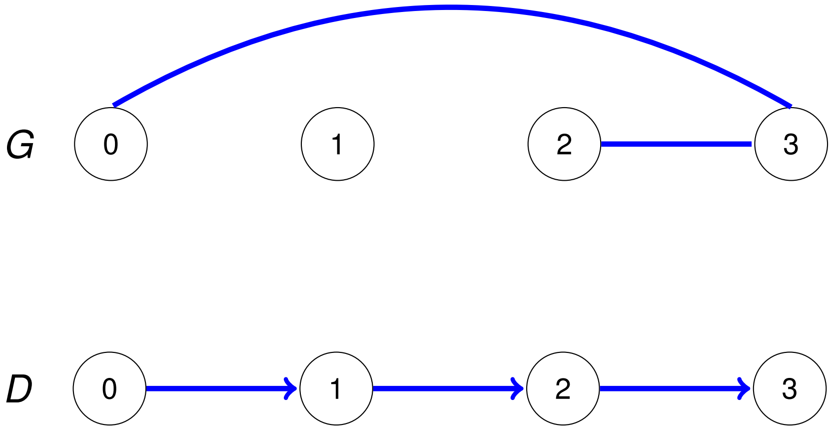
G_1 is not strongly connected and G_4 is not connected.

Recall: Transitive Closure



Recall: Transitive Closure

$E^+ \stackrel{\text{def}}{=}$ smallest **transitive** relation containing E



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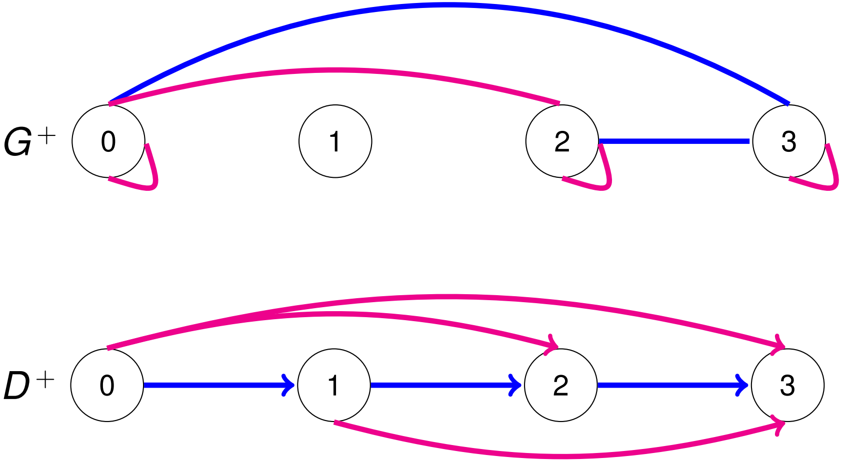
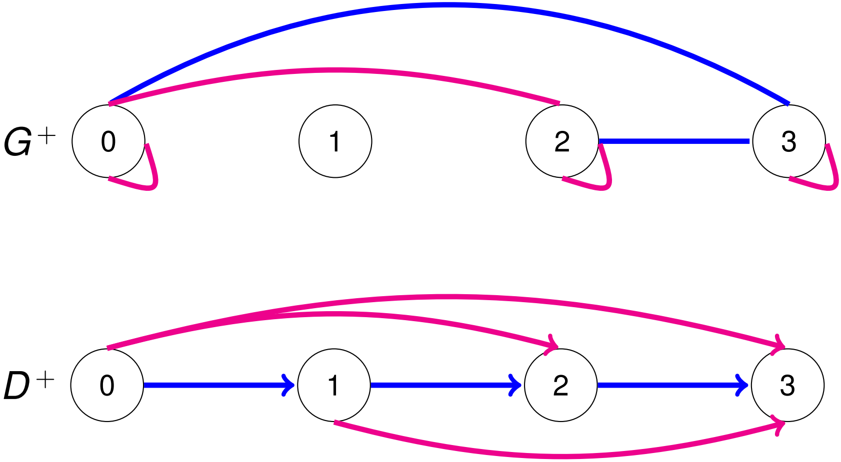


Recall: Transitive Closure

$E^{+} \stackrel{\text{def}}{=}$ smallest **transitive** relation containing E

$E^{\star} \stackrel{\text{def}}{=}$ smallest **reflexive** $\wedge$ **transitive** relation containing E



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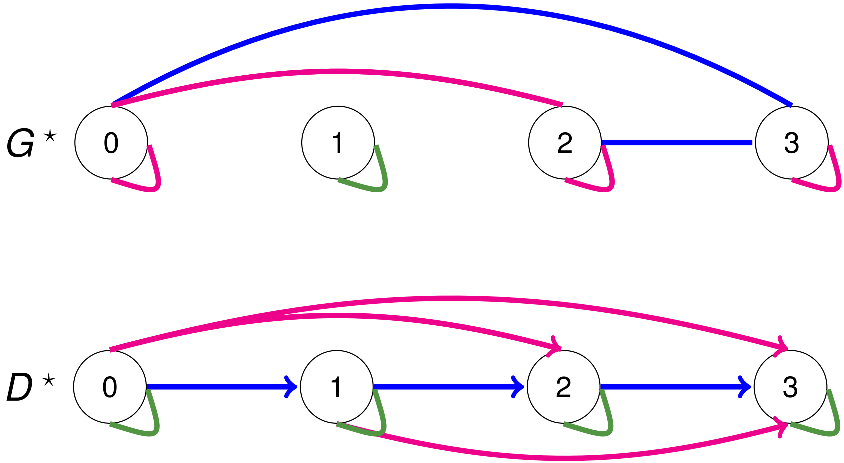
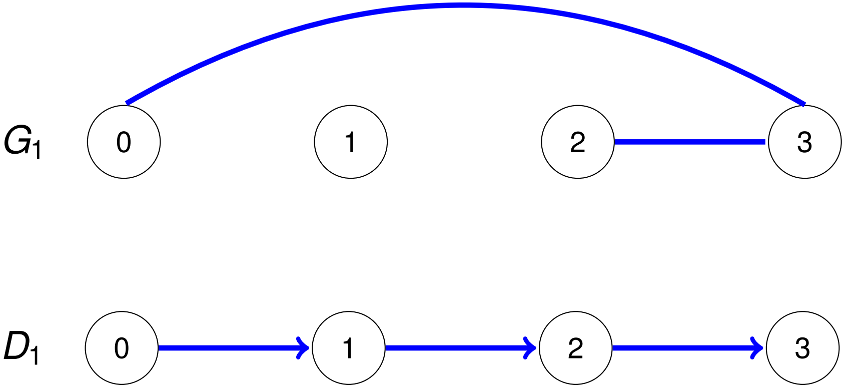


conn $\equiv \forall xy\, E^{\star}(x,y)$

Undirected graph G is **connected** iff $G \models$ **conn**.

Directed graph D is **strongly connected** iff $D \models$ **conn**.



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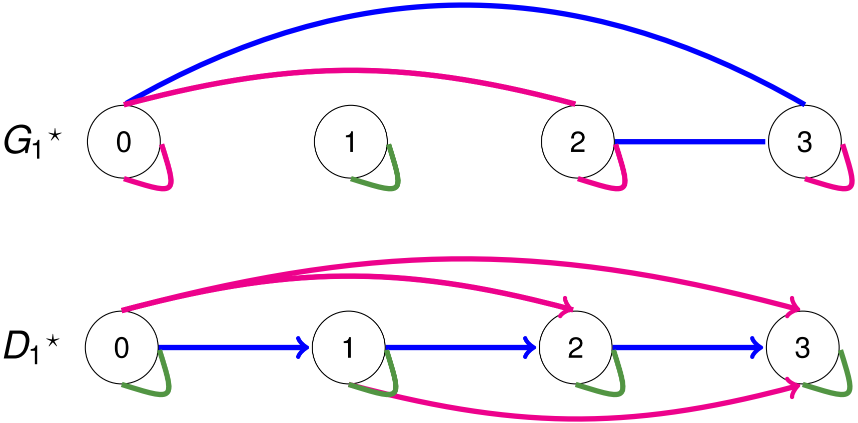
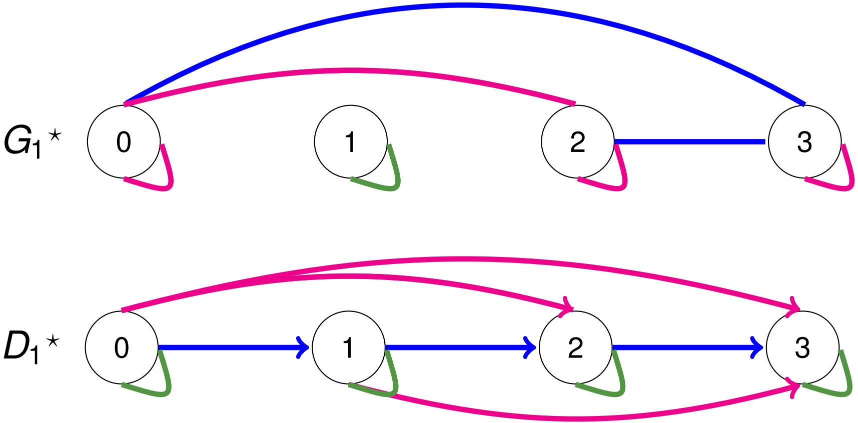


conn $\equiv \forall xy\, E^\star(x,y)$

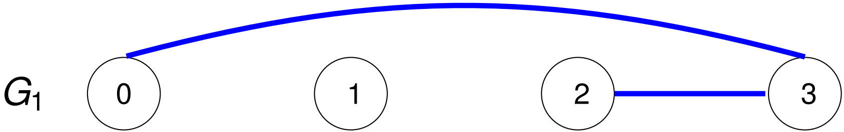
Undirected graph G is **connected** iff $G \models$ **conn**.
G_1 is **not connected**.

Directed graph D is **strongly connected** iff $D \models$ **conn**.
D_1 is **not strongly connected**.



Connected Components

Def: A **connected component** of an undirected graph G is a **maximal induced subgraph** of G that is connected.



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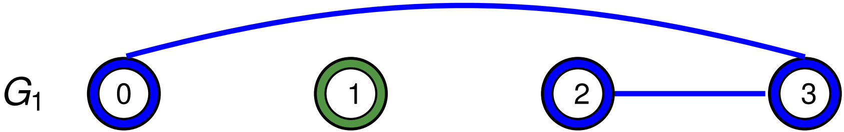

Strongly Connected Components

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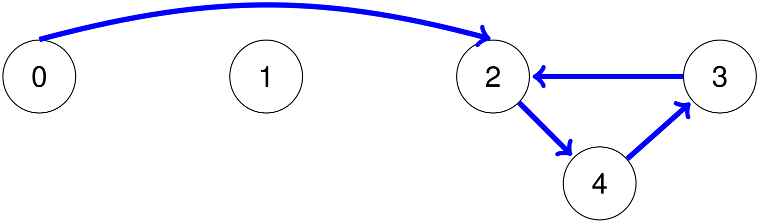
G_1



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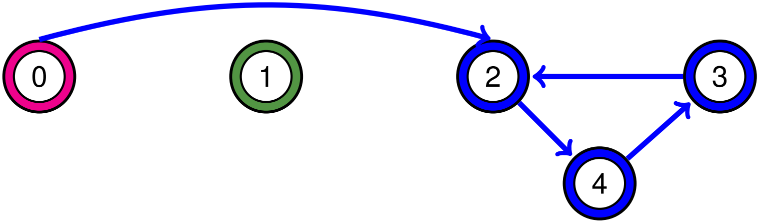
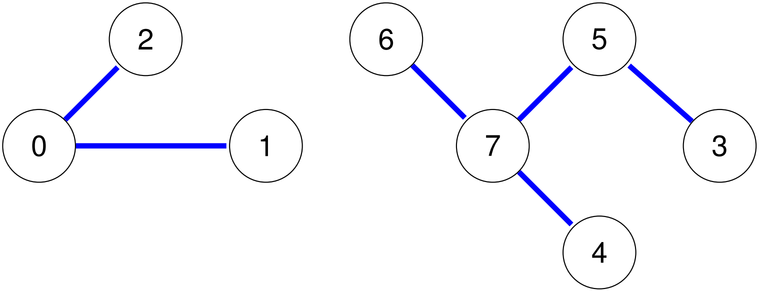


Trees and Forests

Def: An **undirected forest** is an **acyclic undirected graph**

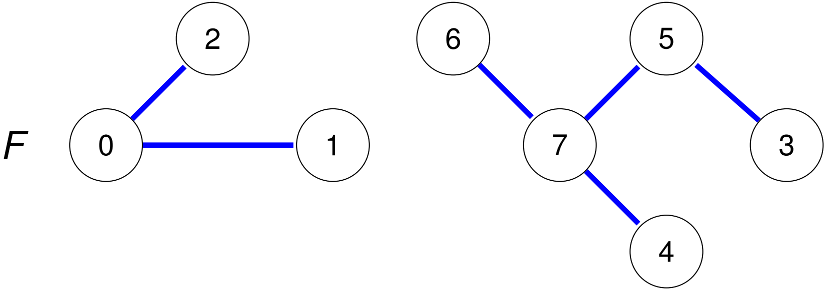
F



Trees and Forests

Def: An **undirected forest** is an **acyclic undirected graph**

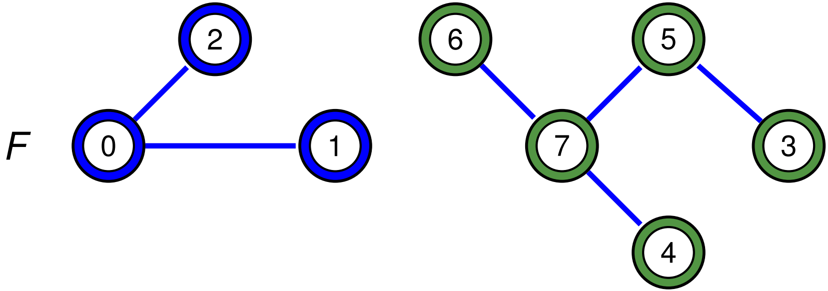
Def: An **undirected tree** is a **connected forest**



Trees and Forests

Def: An **undirected forest** is an **acyclic undirected graph**

Def: An **undirected tree** is a **connected forest**



$F = T_1 \cup T_2$