CS250: Discrete Math for Computer Science

L25: Binary Relations and Digraphs
**Def.** A function $f : A \rightarrow B$ is **one-to-one** ($1 : 1$) iff no element has arrows from two elements: \( \forall xy (f(x) = f(y) \rightarrow x = y) \)
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**Def.** A function \( f : A \rightarrow B \) is **one-to-one** (1:1) iff no element has arrows from two elements: \( \forall xy (f(x) = f(y) \rightarrow x = y) \)

iClicker 25.1 Which functions on the left are 1:1?

A: just \( \text{id}_{[2]} \)

B: just \( g \)

C: both of them

D: neither of them
Def. A function $f : A \to B$ is onto iff every element in $B$ has an arrow to it: \[ \forall y \in B \exists x \in A \quad f(x) = y \]
Def. A function $f : A \rightarrow B$ is **onto** iff every element in $B$ has an arrow to it: $\forall y \in B \ \exists x \in A \ f(x) = y$
**Def.** A function $f : A \rightarrow B$ is **onto** iff every element in $B$ has an arrow to it: $\forall y \in B \ \exists x \in A \ f(x) = y$

\[ id_{[2]}(x) = x \]

\[ g(x) = 1 \]
Def. A function $f : A \to B$ is **onto** iff every element in $B$ has an arrow to it: $\forall y \in B \exists x \in A \ f(x) = y$

iClicker 25.2 Which functions on the left are onto?

- A: just $id_{[2]}$
- B: just $g$
- C: both of them
- D: neither of them
Domain, Range, and Co-Domain

For \( f : A \to B \), its **domain** and **range** are well defined.
Domain, Range, and Co-Domain

For $f : A \to B$, its **domain** and **range** are well defined.

**Def.** The **domain** of $f$: $\text{dom}(f) \overset{\text{def}}{=} \{ a \mid \exists b \ (a, b) \in f \} = A$

For $g : \{1, 2\} \to \{1, 2\}$, its co-domain is $\{1, 2\}$; $g$ is not onto.

For $g : \{1, 2\} \to \{1\}$, its co-domain is $\{1\}$; $g$ is onto.

The co-domain must be given explicitly, it cannot be determined from the function, $g$. A function $g$ is onto iff its range is equal to its co-domain.
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1 \[\rightarrow\] 1

$g = \{(1, 1), (2, 1)\}$

**Def.** The **range** of $f$: $\text{rng}(f) \overset{\text{def}}{=} \{ b \mid \exists a \ (a, b) \in f \}$

2 \[\rightarrow\] 2

$\text{dom}(g) = \{1, 2\}$

$\text{rng}(g) = \{1\}$
For $f : A \rightarrow B$, its **domain** and **range** are well defined.

**Def.** The **domain** of $f$: \( \text{dom}(f) \overset{\text{def}}{=} \{ a \mid \exists b \ (a, b) \in f \} = A \)

\[
\begin{array}{c}
1 \\
\end{array}
\begin{array}{c}
2 \\
\quad g = \{(1, 1), (2, 1)\} \\
\end{array}
\begin{array}{c}
1 \\
\end{array}
\begin{array}{c}
2 \\
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\end{array}
\]

**Def.** The **range** of $f$: \( \text{rng}(f) \overset{\text{def}}{=} \{ b \mid \exists a \ (a, b) \in f \} \)

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**Def.** The **domain** of $f$: $\text{dom}(f) \overset{\text{def}}{=} \{ a \mid \exists b \ (a, b) \in f \} = A$

1 \rightarrow 1 \quad \text{dom}(g) = \{1, 2\}

$g = \{(1, 1), (2, 1)\}$

2 \rightarrow 1 \quad \text{rng}(g) = \{1\}

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$1 \quad 1$

$2 \quad g = \{(1, 1), (2, 1)\} \quad 2$

dom($g$) = \{1, 2\}

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```
1
   ___  ___  ___  ___
(1)  -  (2)  -  (1)
   ___  ___  ___  ___
2
```

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A function \( g \) is **onto** iff its **range** is equal to its **co-domain**.

For \( f \subseteq A \times B \), we can tell if \( f \) is **single valued** and if it is **1:1**.
For $f : A \rightarrow B$, its **domain** and **range** are well defined.

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For \( f \subseteq A \times B \), we can tell if \( f \) is **single valued** and if it is **1:1**.

To tell if \( f : A \to B \), i.e., is \( f \) a **function**, we must know \( A \).

To tell if \( f \) is **onto**, we must know \( B \).
Composition of Functions

For \( f : A \rightarrow B \) and \( g : B \rightarrow C \),

\[
(g \circ f)(x) = g(f(x))
\]

\[
f \circ g : N \rightarrow N:
f \circ g(n) = 2 \cdot n + 1
\]
Composition of Functions

For $f : A \to B$ and $g : B \to C$,

**Def.** the **composition** of $g$ and $f$: \[ g \circ f(x) \stackrel{\text{def}}{=} g(f(x)) \]
For $f : A \rightarrow B$ and $g : B \rightarrow C$,

**Def.** the **composition** of $g$ and $f$:  
$g \circ f(x) \overset{\text{def}}{=} g(f(x))$

$f : \mathbb{N} \rightarrow \mathbb{N}$:  
$f(n) = 2 \cdot n$  
$g : \mathbb{N} \rightarrow \mathbb{N}$:  
$g(n) = n + 1$
Composition of Functions

For $f : A \rightarrow B$ and $g : B \rightarrow C$,

**Def.** the composition of $g$ and $f$: $g \circ f(x) \overset{\text{def}}{=} g(f(x))$

$f : \mathbb{N} \rightarrow \mathbb{N} : f(n) = 2 \cdot n$

$g : \mathbb{N} \rightarrow \mathbb{N} : g(n) = n + 1$

$g \circ f : \mathbb{N} \rightarrow \mathbb{N} : g \circ f(n) = g(f(n)) = 2 \cdot n + 1$
Composition of Functions

For $f : A \rightarrow B$ and $g : B \rightarrow C$,

**Def.** the **composition** of $g$ and $f$: \( g \circ f(x) \overset{\text{def}}{=} g(f(x)) \)

- $f : \mathbb{N} \rightarrow \mathbb{N}$: \( f(n) = 2 \cdot n \)
- $g : \mathbb{N} \rightarrow \mathbb{N}$: \( g(n) = n + 1 \)

- $(g \circ f) : \mathbb{N} \rightarrow \mathbb{N}$: \( g \circ f(n) = g(f(n)) = 2 \cdot n + 1 \)

- $(f \circ g) : \mathbb{N} \rightarrow \mathbb{N}$: \( f \circ g(n) = f(g(n)) = 2 \cdot (n + 1) \)
Inverse of functions

For $f : A \rightarrow B$ and $g : B \rightarrow A$, 

$f_1 : \mathbb{Z} \rightarrow \mathbb{Z}$ 
$f_1(x) = x + 1$; 
$g_1 : \mathbb{Z} \rightarrow \mathbb{Z}$ 
$g_1(x) = x - 1$; 

$f_1 \circ g_1(x) = f_1(g_1(x)) = (x - 1) + 1 = x$ 
$g_1 \circ f_1(x) = g_1(f_1(x)) = (x + 1) - 1 = x$ 

$f_2 : \mathbb{Q} \rightarrow \mathbb{Q}$ 
$f_2(x) = x \cdot 2$; 
$g_2 : \mathbb{Q} \rightarrow \mathbb{Q}$ 
$g_2(x) = x / 2$; 

$f_2 \circ g_2(x) = f_2(g_2(x)) = (x / 2) \cdot 2 = x$ 
$g_2 \circ f_2(x) = g_2(f_2(x)) = (x \cdot 2) / 2 = x$
For $f : A \to B$ and $g : B \to A$, 

**Def.** $f$ and $g$ are **inverse functions** $f = g^{-1}$ and $g = f^{-1}$ iff 

$$f \circ g = \text{id}_B; \quad \text{and} \quad g \circ f = \text{id}_A$$
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\( f_1 : \mathbb{Z} \rightarrow \mathbb{Z} \quad f_1(x) = x + 1; \quad g_1 : \mathbb{Z} \rightarrow \mathbb{Z} \quad g_1(x) = x - 1 \)
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\[
f_1 \circ g_1(x) = f_1(g_1(x)) = (x - 1) + 1 = x
\]
Inverse of functions

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$$f_1 \circ g_1(x) = f_1(g_1(x)) = (x - 1) + 1 = x$$

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$$g_1 \circ f_1(x) = g_1(f_1(x)) = (x + 1) - 1 = x$$

$f_2 : \mathbb{Q} \rightarrow \mathbb{Q} \quad f_2(x) = x \cdot 2; \quad g_2 : \mathbb{Q} \rightarrow \mathbb{Q} \quad g_2(x) = x/2$
Inverse of functions

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**Def.** \( f \) and \( g \) are **inverse functions** \( f = g^{-1} \) and \( g = f^{-1} \) iff

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\[
f_2 \circ g_2(x) = f_2(g_2(x)) = (x/2) \cdot 2 = x
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Inverse of functions

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\( f_2 : \mathbb{Q} \to \mathbb{Q} \quad f_2(x) = x \cdot 2; \quad g_2 : \mathbb{Q} \to \mathbb{Q} \quad g_2(x) = x/2 \)

\[
f_2 \circ g_2(x) = f_2(g_2(x)) = (x/2) \cdot 2 = x
\]

\[
g_2 \circ f_2(x) = g_2(f_2(x)) = (x \cdot 2)/2 = x
\]
For $f : A \rightarrow B$ and $g : B \rightarrow A$,

$f$ and $g$ are inverse functions $f = g^{-1}$ and $g = f^{-1}$ iff

$f(x) = (x + 1) \% 3$
Inverse of functions

For $f : A \to B$ and $g : B \to A$, $f$ and $g$ are inverse functions $f = g^{-1}$ and $g = f^{-1}$ iff $f \circ g = \text{id}_B$

$g(x) = (x + 2) \% 3$  \hspace{1cm} $f(x) = (x + 1) \% 3$
Inverse of functions

For $f : A \to B$ and $g : B \to A$,

$f$ and $g$ are **inverse functions** $f = g^{-1}$ and $g = f^{-1}$ iff

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\[
g(x) = (x + 2) \% 3 \quad f(x) = (x + 1) \% 3 \quad g(x) = (x + 2) \% 3
\]
When does $f : A \rightarrow B$ have an inverse?

Does $f_1 : \mathbb{N} \rightarrow \mathbb{N}$, $f_1(n) = \lfloor n/2 \rfloor$ have an inverse?
When does $f : A \rightarrow B$ have an inverse?

Does $f_1 : \mathbb{N} \rightarrow \mathbb{N}, \quad f_1(n) = \lfloor n/2 \rfloor$ have an inverse?

No, $f_1$ is not 1:1, so no $g_1$ cannot satisfy $g_1(f_1(0)) = 0$ and $g_1(f_1(1)) = 1$ because $f_1(0) = f_1(1)$. 

Thm. $f : A \rightarrow B$ has an inverse iff $f$ is 1:1 and onto.

Proof: Already argued it is necessary that $f$ is 1:1 and onto. Assume that $f$ is 1:1 and onto. Let $f^T \text{def}= \{ (b, a) \mid (a, b) \in f \}$ transpose of $f$. $f^T : B \rightarrow A$ and $f^T \circ f = \text{id}_A$ and $f \circ f^T = \text{id}_B \Box$
When does \( f : A \rightarrow B \) have an inverse?

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Does \( f_2 : \mathbb{N} \rightarrow \mathbb{N}, \quad f_2(n) = n + 1 \) have an inverse?
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Does \( f_2 : \mathbb{N} \rightarrow \mathbb{N}, \quad f_2(n) = n + 1 \) have an inverse?

No, \( f_2 \) is not onto, so no \( g_2 \) can satisfy \( f_2(g_2(0)) = 0 \) because \( 0 \not\in \text{rng}(f_2) \).
When does \( f : A \rightarrow B \) have an inverse?

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When does \( f : A \rightarrow B \) have an inverse?

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**Proof:** Already argued it is necessary that \( f \) is 1:1 and onto.
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**Proof:** Already argued it is necessary that $f$ is 1:1 and onto.

Assume that $f$ is 1:1 and onto.
When does $f : A \to B$ have an inverse?

Does $f_1 : \mathbb{N} \to \mathbb{N}, \quad f_1(n) = \lfloor n/2 \rfloor$ have an inverse?

No, $f_1$ is not 1:1, so no $g_1$ cannot satisfy $g_1(f_1(0)) = 0$ and $g_1(f_1(1)) = 1$ because $f_1(0) = f_1(1)$.

Does $f_2 : \mathbb{N} \to \mathbb{N}, \quad f_2(n) = n + 1$ have an inverse?

No, $f_2$ is not onto, so no $g_2$ can satisfy $f_2(g_2(0)) = 0$ because $0 \notin \text{rng}(f_2)$

**Thm.** $f : A \to B$ has an inverse iff $f$ is 1:1 and onto.

**Proof:** Already argued it is necessary that $f$ is 1:1 and onto.

Assume that $f$ is 1:1 and onto.

Let $f^T \overset{\text{def}}{=} \{(b, a) \mid (a, b) \in f\}$ transpose of $f$. 
When does $f : A \rightarrow B$ have an inverse?

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$f^T : B \rightarrow A$ and $f^T \circ f = \text{id}_A$ and $f \circ f^T = \text{id}_B$.
Claim: $f \circ f^T = \text{id}_R$ and $f^T \circ f = \text{id}_A$

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Claim: \[ f \circ f^T = \text{id}_R \quad \text{and} \quad f^T \circ f = \text{id}_A \]

Proof: \[ f \text{ is onto: } \forall y \in R \ \exists x \in A \ (y, x) \in f^T, \text{ thus } f \circ f^T(y) = y \]
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Proof: \( f \) is onto: \( \forall y \in R \exists x \in A (y, x) \in f^T \), thus \( f \circ f^T(y) = y \)

\( f \) is 1:1: \( \forall x \in A \ f^T \circ f(x) = x \) \qquad \Box
Def. A directed graph (digraph), $G = (V^G, E^G)$ is a world of vocabulary $\Sigma_g = (E^2; )$. Thus a digraph, $G$, is just a binary relation, $E^G$, from $V^G$ to $V^G$. 
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![Diagram of a directed graph with nodes 1, 2, and 3 connected by directed edges]
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![Diagram of directed graphs]
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\[ 1 \leq [3] \]

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\[ 1 \equiv (\text{mod } 2) \]
reflexive \equiv \forall x \ E(x, x)

\[
\begin{array}{c}
\equiv \left[3\right] \\
1 & \rightarrow & 2 & \rightarrow & 3 \\
\equiv \left[3\right] \\
\end{array}
\quad
\begin{array}{c}
\equiv \left[3\right] \\
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\end{array}
\]
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symmetric \equiv \forall xy \ (E(x, y) \rightarrow E(y, x))
reflexive \equiv \forall x \; E(x, x)

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transitive \equiv \forall xyz \; (E(x, y) \land E(y, z) \rightarrow E(x, z))
reflexive  ≡ ∀x \ E(x, x)

symmetric  ≡ ∀xy (E(x, y) → E(y, x))

transitive  ≡ ∀xyz (E(x, y) ∧ E(y, z) → E(x, z))

Which are Reflexive, Symmetric and Transitive?

A: all
B: just \equiv (\text{mod } 2)
C: \equiv^{[3]} \text{ and } \equiv (\text{mod } 2)
D: all but <^{[3]}
**Def. Transitive Closure** $E^+$ is the smallest *transitive* relation containing $E$.

![Diagram](attachment:image.png)
**Def. Transitive Closure** $E^+$ is the smallest **transitive** relation containing $E$.
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Connectivity

\[ \text{conn} \equiv \forall xy \ E^*(x, y) \]

Undirected graph \( G \) is **connected** iff \( G \models \text{conn} \).

Directed graph \( G \) is **strongly connected** iff \( G \models \text{conn} \).

\( G_1 \) is not strongly connected and \( G_4 \) is not connected.
Recall: Transitive Closure

\[ E^+ = \text{smallest transitive relation containing } E \]

\[ E^* = \text{smallest reflexive } \land \text{transitive relation containing } E \]
Recall: Transitive Closure

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Recall: Transitive Closure

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Undirected graph \( G \) is **connected** iff \( G \models \text{conn} \).

Directed graph \( D \) is **strongly connected** iff \( D \models \text{conn} \).
\[ \text{conn} \equiv \forall xy \ E^*(x, y) \]

Undirected graph \( G \) is \textit{connected} iff \( G \models \text{conn} \).

Directed graph \( D \) is \textit{strongly connected} iff \( D \models \text{conn} \).
\text{conn} \equiv \forall xy \ E^*(x, y)

Undirected graph \( G \) is \textbf{connected} iff \( G \models \text{conn} \).
\( G_1 \) is \textbf{not connected}.

Directed graph \( D \) is \textbf{strongly connected} iff \( D \models \text{conn} \).
\( D_1 \) is \textbf{not strongly connected}.

\begin{figure}[h]
\begin{center}
\begin{tikzpicture}
    \node (0) at (0,0) {0};
    \node (1) at (2,0) {1};
    \node (2) at (4,0) {2};
    \node (3) at (6,0) {3};
    \draw[blue, thick] (0) to (3);
    \draw[blue, thick] (3) to (2);
    \draw[blue, thick] (2) to (1);
    \draw[blue, thick] (1) to (0);
    \end{tikzpicture}
\end{center}
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    \draw[blue, thick] (2) to (1);
    \draw[blue, thick] (1) to (0);
    \draw[purple, thick] (0) to (1);
    \draw[purple, thick] (1) to (2);
    \draw[purple, thick] (2) to (3);
    \draw[purple, thick] (3) to (0);
    \draw[green, thick, bend right=45] (0) to (1);
    \draw[green, thick, bend right=45] (1) to (2);
    \draw[green, thick, bend right=45] (2) to (3);
    \draw[green, thick, bend right=45] (3) to (0);
    \end{tikzpicture}
\end{center}
\end{figure}
Def: A connected component of an undirected graph $G$ is a maximal induced subgraph of $G$ that is connected.
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Def: An undirected forest is an acyclic undirected graph
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Def: An undirected tree is a connected forest
Def: An undirected forest is an acyclic undirected graph

Def: An undirected tree is a connected forest

\[ F = T_1 \cup T_2 \]