L24: Connectivity in Graphs and Digraphs
A path is a walk that never visits the same edge or vertex twice. [Rosen calls paths “simple paths”]

Note: this definition differs for directed and undirected graphs. The undirected graph, $G_1$, below is acyclic. However, the directed graph, $D_1$, has a cycle: $(0, 1, 0)$. This is because in the undirected graph, the edges $(0, 1)$ and $(1, 0)$ are considered the same edge, so a path may not traverse this edge twice. However, they are different edges in the directed graph.
A **path** is a **walk** that **never visits the same edge or vertex twice**. [Rosen calls paths “simple paths”] except a path may **start and end** at the **same vertex** in which case it is called a **cycle**.

**Note:** this definition differs for directed and undirected graphs. The undirected graph, $G_1$, below is acyclic. However, the directed graph, $D_1$, has a cycle: $(0, 1, 0)$. This is because in the undirected graph, the edges $(0, 1)$ and $(1, 0)$ are considered the same edge, so a path may not traverse this edge twice. However, they are different edges in the directed graph.
A path is a walk that never visits the same edge or vertex twice. [Rosen calls paths “simple paths”] except a path may start and end at the same vertex in which case it is called a cycle. A loop is a cycle of length 1. There are no cycles of length 0.

Note: this definition differs for directed and undirected graphs. The undirected graph, $G_1$, below is acyclic. However, the directed graph, $D_1$, has a cycle: $(0, 1, 0)$. This is because in the undirected graph, the edges $(0, 1)$ and $(1, 0)$ are considered the same edge, so a path may not traverse this edge twice. However, they are different edges in the directed graph.

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$w_1 = (0)$  \hspace{1cm} w_2 = (0, 1)$  \hspace{1cm} w_3 = (0, 1, 2)$

$w_4 = (0, 1, 0)$  \hspace{1cm} w_5 = (0, 1, 2, 3)$  \hspace{1cm} w_6 = (0, 1, 2, 3, 0)$
In the above undirected graph, which of the above walks are paths?

A: all of them  B: all except $w_4$  C: all except $w_4$ and $w_6$
\( w_1 = (0) \quad w_2 = (0, 1) \quad w_3 = (0, 1, 2) \)
\( w_4 = (0, 1, 0) \quad w_5 = (0, 1, 2, 3) \quad w_6 = (0, 1, 2, 3, 0) \)

**iClicker:** In the above undirected graph, which of the above walks are paths?

A: all of them  
B: all except \( w_4 \)  
C: all except \( w_4 \) and \( w_6 \)

**iClicker:** which of the above walks are cycles?

A: \( w_4 \) and \( w_6 \)  
B: just \( w_6 \)
Cyclic versus Acyclic

A graph that has \textbf{at least one cycle} is called \textbf{cyclic}.

\begin{itemize}
  \item \textbf{G}₁
  \begin{tikzpicture}
    \node (s) at (0,0) {$s$};
    \node (1) at (1,0) {1};
    \node (2) at (2,0) {2};
    \node (3) at (3,0) {3};
    \node (t) at (4,0) {$t$};
    \path (s) edge (1);
    \path (1) edge (2);
    \path (2) edge (3);
    \path (3) edge (t);
  \end{tikzpicture}

  \item \textbf{G}₂
  \begin{tikzpicture}
    \node (0) at (0,0) {0};
    \node (1) at (1,0) {1};
    \node (2) at (2,0) {2};
    \node (3) at (3,0) {3};
    \node (t) at (4,0) {$t$};
    \path (0) edge (1);
    \path (1) edge (2);
    \path (2) edge (3);
    \path (3) edge[bend left] (0);
    \path (t) edge[bend right] (0);
  \end{tikzpicture}

  \item \textbf{G}₃
  \begin{tikzpicture}
    \node (0) at (0,0) {0};
    \node (1) at (1,0) {1};
    \node (2) at (2,0) {2};
    \node (3) at (3,0) {3};
    \node (s) at (4,0) {$s$};
    \node (t) at (5,0) {$t$};
    \path (s) edge (0);
    \path (0) edge (1);
    \path (1) edge (2);
    \path (2) edge (3);
    \path (3) edge (s);
    \path (3) edge (t);
  \end{tikzpicture}

  \item \textbf{G}₄
  \begin{tikzpicture}
    \node (0) at (0,0) {0};
    \node (1) at (1,0) {1};
    \node (2) at (2,0) {2};
    \node (3) at (3,0) {3};
    \node (s) at (4,0) {$s$};
    \node (t) at (5,0) {$t$};
    \path (s) edge (0);
    \path (0) edge (1);
    \path (1) edge (2);
    \path (2) edge (3);
    \path (3) edge (s);
    \path (3) edge (t);
  \end{tikzpicture}
\end{itemize}
Cyclic versus Acyclic

A graph that has **at least one cycle** is called **cyclic**.

A graph that has **no cycles** is called **acyclic**.
Cyclic versus Acyclic

A graph that has **at least one cycle** is called **cyclic**.

A graph that has **no cycles** is called **acyclic**.

\[ G_1, G_2, G_3, G_4 \]

**iClicker**: which of the above graphs are acyclic?

**A**: all of them      **B**: all except \( G_3 \)      **C**: all except \( G_2 \) and \( G_3 \)
A graph that has **at least one cycle** is called **cyclic**.

A graph that has **no cycles** is called **acyclic**.

\[ \text{G}_1 \quad \text{G}_2 \quad \text{G}_3 \quad \text{G}_4 \]

**iClicker:** which of the above graphs are acyclic?

**A:** all of them  **B:** all except \( \text{G}_3 \)  **C:** all except \( \text{G}_2 \) and \( \text{G}_3 \)

To emphasize that it might not be undirected, we sometimes call a directed graph a **digraph**.
A graph that has at least one cycle is called cyclic.
A graph that has no cycles is called acyclic.

\[ G_1 \quad G_2 \quad G_3 \quad G_4 \]

**iClicker:** which of the above graphs are acyclic?

**A:** all of them    **B:** all except \( G_3 \)    **C:** all except \( G_2 \) and \( G_3 \)

To emphasize that it might not be undirected, we sometimes call a directed graph a digraph. A Directed Acyclic Graph is called a DAG.
reflexive \equiv \forall x \ E(x, x)

symmetric \equiv \forall xy \ (E(x, y) \rightarrow E(y, x))

transitive \equiv \forall xyz \ (E(x, y) \land E(y, z) \rightarrow E(x, z))
reflexive \equiv \forall x \ E(x, x)

symmetric \equiv \forall xy \ (E(x, y) \rightarrow E(y, x))

transitive \equiv \forall xyz \ (E(x, y) \land E(y, z) \rightarrow E(x, z))

G_1

G_2

G_3

G_4
reflexive  $\equiv \forall x \ E(x, x)$

symmetric $\equiv \forall xy \ (E(x, y) \rightarrow E(y, x))$

transitive $\equiv \forall xyz \ (E(x, y) \land E(y, z) \rightarrow E(x, z))$

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G_1

G_2

G_3

G_4

---

iClicker: which graphs above are reflexive?, $G \models \text{reflexive}$

A: none of them  B: just $G_3$  C: just $G_2$ and $G_3$
reflexive \equiv \forall x \ E(x, x)

symmetric \equiv \forall xy \ (E(x, y) \to E(y, x))

transitive \equiv \forall xyz \ (E(x, y) \land E(y, z) \to E(x, z))
reflexive \equiv \forall x \ E(x, x) \\
symmetric \equiv \forall xy \ (E(x, y) \rightarrow E(y, x)) \\
transitive \equiv \forall xyz \ (E(x, y) \wedge E(y, z) \rightarrow E(x, z))

\begin{itemize}
  \item \textbf{iClicker:} which graphs above are reflexive? \hspace{1cm} G \models \text{reflexive}
    \begin{enumerate}
      \item A: none of them
      \item B: just $G_3$
      \item C: just $G_2$ and $G_3$
    \end{enumerate}
  \item \textbf{iClicker:} which are symmetric? \hspace{1cm} G \models \text{symmetric}
    \begin{enumerate}
      \item A: none of them
      \item B: just $G_3$
      \item C: just $G_3$ and $G_4$
    \end{enumerate}
  \item \textbf{iClicker:} which are transitive? \hspace{1cm} G \models \text{transitive}
    \begin{enumerate}
      \item A: none of them
      \item B: just $G_3$
      \item C: all but $G_4$
    \end{enumerate}
\end{itemize}
Transitive Closure

$G_1$ $G_4$

$L24$: Connectivity in Graphs and Digraphs  
CS250: Discrete Math for Computer Science
Transitive Closure

\[ E^+ \overset{\text{def}}{=} \text{smallest transitive relation containing } E \]
Transitive Closure

\[ E^+ \overset{\text{def}}{=} \text{smallest transitive relation containing } E \]

Graphs:

- \( G_1^+ \)
  - Vertices: \( s, t \)
  - Edges: \( s \rightarrow t \)

- \( G_4^+ \)
  - Vertices: \( 0, 1, 2, 3, s, t \)
  - Edges: \( s \rightarrow t, 0 \rightarrow 1, 1 \rightarrow 2, 2 \rightarrow 3 \)
Transitive Closure

\[ E^+ \overset{\text{def}}{=} \text{smallest transitive relation containing } E \]

\[ E^* \overset{\text{def}}{=} \text{smallest reflexive } \land \text{ transitive relation containing } E \]
Transitive Closure

\[ E^+ \overset{\text{def}}{=} \text{smallest transitive relation containing } E \]

\[ E^* \overset{\text{def}}{=} \text{smallest reflexive } \land \text{ transitive relation containing } E \]
Connectivity

\[
\text{conn} \equiv \forall xy \ E^*(x, y)
\]

Undirected graph \( G \) is **connected** iff \( G \models \text{conn} \).

Directed graph \( G \) is **strongly connected** iff \( G \models \text{conn} \).

\[ G_1 \] is not strongly connected and \( G_4 \) is not connected.
Recall: Transitive Closure

\[G\]

\[D\]
Recall: Transitive Closure

\[ E^+ \overset{\text{def}}{=} \text{smallest transitive relation containing } E \]
Recall: Transitive Closure

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Recall: Transitive Closure

\[ E^+ \overset{\text{def}}{=} \text{smallest transitive relation containing } E \]

\[ E^* \overset{\text{def}}{=} \text{smallest reflexive } \land \text{ transitive relation containing } E \]
Recall: Transitive Closure

\[ E^+ \overset{\text{def}}{=} \text{smallest transitive relation containing } E \]

\[ E^* \overset{\text{def}}{=} \text{smallest reflexive } \wedge \text{ transitive relation containing } E \]

Diagram of \( G^* \)

Diagram of \( D^* \)
\[ \text{conn} \equiv \forall xy E^*(x, y) \]

Undirected graph $G$ is \textbf{connected} iff $G \models \text{conn}$.

Directed graph $D$ is \textbf{strongly connected} iff $D \models \text{conn}$.
\[ \text{conn} \equiv \forall xy E^*(x, y) \]

Undirected graph \( G \) is \textit{connected} iff \( G \models \text{conn} \).

Directed graph \( D \) is \textit{strongly connected} iff \( D \models \text{conn} \).
conn \equiv \forall xy E^*(x, y)

Undirected graph \( G \) is connected iff \( G \models \text{conn} \).
\( G_1 \) is not connected.

Directed graph \( D \) is strongly connected iff \( D \models \text{conn} \).
\( D_1 \) is not strongly connected.
**Def:** A connected component of an undirected graph $G$ is a maximal induced subgraph of $G$ that is connected.
Def: A connected component of an undirected graph $G$ is a maximal induced subgraph of $G$ that is connected.
**Def:** A strongly connected component of a directed graph $G$ is a maximal induced subgraph of $G$ that is strongly connected.
**Def:** A strongly connected component of a directed graph $G$ is a maximal induced subgraph of $G$ that is strongly connected.

$$G_1$$

$0$ $1$ $2$ $3$ $4$
Def: An undirected forest is an acyclic undirected graph
Def: An undirected forest is an acyclic undirected graph

Def: An undirected tree is a connected forest
Def: An undirected forest is an acyclic undirected graph

Def: An undirected tree is a connected forest

\[ F = T_1 \cup T_2 \]