CS250: Discrete Math for Computer Science

L23: Truth Game Wrap Up

Tarski's Recursive Definition of Truth

$$G(t_1 = t_2) \stackrel{\text{def}}{=} t_1^G == t_2^G$$

$$G(P(t_1, \ldots, t_a)) \stackrel{\text{def}}{=} (t_1^G, \ldots, t_a^G) \in P^G$$

$$G(\sim\alpha) \stackrel{\text{def}}{=} 1 - G(\alpha)$$

$$G(\alpha \wedge \beta) \stackrel{\text{def}}{=} \min(G(\alpha), G(\beta))$$

$$G(\alpha \vee \beta) \stackrel{\text{def}}{=} \max(G(\alpha), G(\beta))$$

$$G(\forall x(\alpha)) \stackrel{\text{def}}{=} \min_{a \in |G|} G[a/x](\alpha)$$

$$G(\exists x(\alpha)) \stackrel{\text{def}}{=} \max_{a \in |G|} G[a/x](\alpha)$$

Truth Game: a two player game that is an equivalent but more fun way to tell whether $W \models \varphi$. First put φ into NNF.

Dumbledore wants to show that $W \models \varphi$

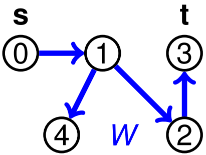
Gandalf wants to show that $W \not\models \varphi$.

base case: if φ is a literal, then **D** wins iff $W \models \varphi$.
inductive cases:

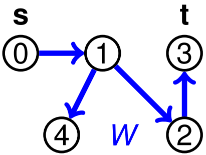
$W \models \varphi \wedge \psi$	**G** chooses $\alpha \in \{\varphi, \psi\}$	continue on: $W \models \alpha$
$W \models \varphi \vee \psi$	**D** chooses $\alpha \in \{\varphi, \psi\}$	continue on: $W \models \alpha$
$W \models \forall x\, \varphi$	**G** chooses $a \in \lvert W \rvert$	continue on: $W\, a/x \models \varphi$
$W \models \exists x\, \varphi$	**D** chooses $a \in \lvert W \rvert$	continue on: $W\, a/x \models \varphi$

Truth Game Example

Does $W \models \forall x\,(x = s \vee \exists y\, E(y, x))$?



Truth Game Example



Does $W \models \forall x\,(x = s \vee \exists y\, E(y, x))$?

G moves, chooses $x = 4$

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Does $W \models \forall x\ (x = s \vee \exists y\ E(y, x))$?

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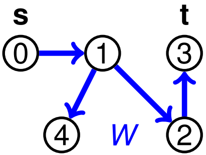
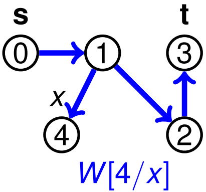


Does $W[4/x] \models x = s \vee \exists y\ E(y, x)$?



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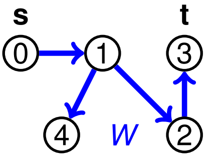
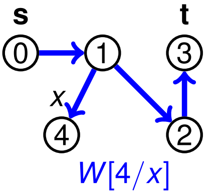


Does $W[4/x] \models x = s \vee \exists y\, E(y, x)$?

D moves, chooses $\exists y\, E(y, x)$



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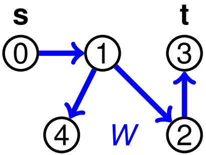

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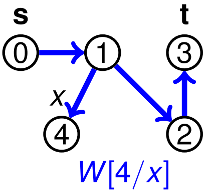
D moves, chooses $\exists y\ E(y, x)$



Does $W[4/x] \models \exists y\ E(y, x)$?

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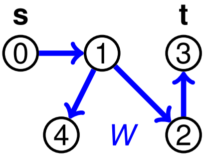
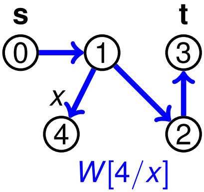


Does $W[4/x] \models x = s \vee \exists y\ E(y,x)$?

D moves, chooses $\exists y\ E(y,x)$

Does $W[4/x] \models \exists y\ E(y,x)$?

D moves, chooses $y = 1$



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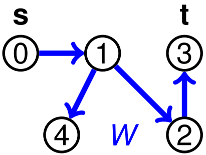
G moves, chooses $x = 4$



Does $W[4/x] \models x = s \vee \exists y\, E(y,x)$?

D moves, chooses $\exists y\, E(y,x)$

Does $W[4/x] \models \exists y\, E(y,x)$?

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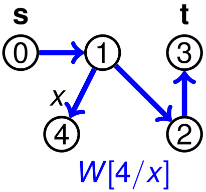
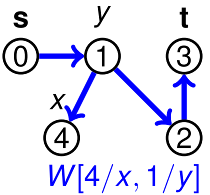


Does $W[4/x, 1/y] \models E(y,x)$?



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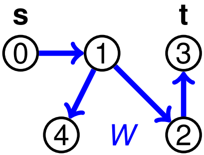
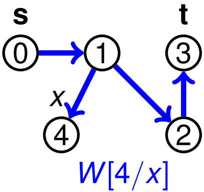


Does $W[4/x] \models x = s \vee \exists y\, E(y,x)$?

D moves, chooses $\exists y\, E(y,x)$

Does $W[4/x] \models \exists y\, E(y,x)$?

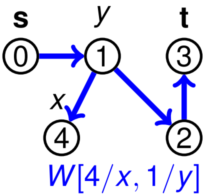
D moves, chooses $y = 1$



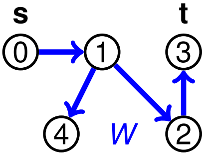
Does $W[4/x, 1/y] \models E(y,x)$?

Yes, **D** wins!

$W \models \forall x\,(x = s \vee \exists y\, E(y,x))$

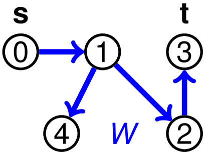


Game Quiz: Make **first winning choice** else **0** or first choice if there is none.



Game Quiz: Make **first winning choice** else **0** or first choice if there is none.

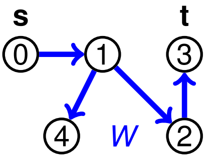
Does $W \models \forall x \, (x = s \lor \exists y \, E(y, x))$?

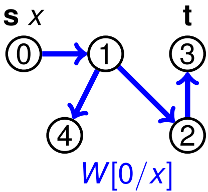


Game Quiz: Make **first winning choice** else **0** or first choice if there is none.

Does $W \models \forall x \, (x = s \vee \exists y \, E(y, x))$?

G moves, chooses $x = 0$





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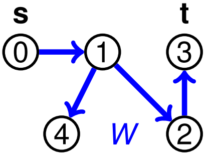

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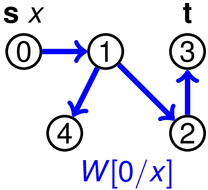
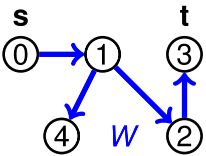


Game Quiz: Make **first winning choice**
else **0** or first choice if there is none.

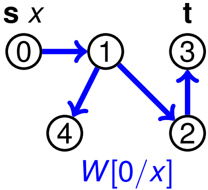


Does $W \models \forall x\ (x = s \vee \exists y\ E(y, x))$?

G moves, chooses $x = 0$

Does $W[0/x] \models x = s \vee \exists y\ E(y, x)$?

D moves, chooses $x = s$



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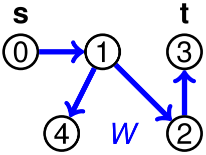
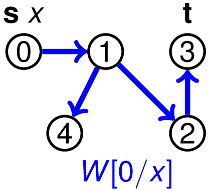


Does $W \models \forall x\ (x = s \vee \exists y\ E(y, x))$?

G moves, chooses $x = 0$

Does $W[0/x] \models x = s \vee \exists y\ E(y, x)$?

D moves, chooses $x = s$



Does $W[0/x] \models x = s$?

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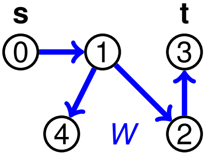
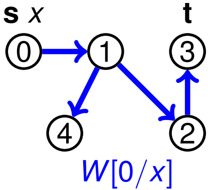


Does $W \models \forall x\ (x = s \vee \exists y\ E(y, x))$?

G moves, chooses $x = 0$

Does $W[0/x] \models x = s \vee \exists y\ E(y, x)$?

D moves, chooses $x = s$



Does $W[0/x] \models x = s$?

Yes, **D** wins!

$W \models \forall x\ (x = s \vee \exists y\ E(y, x))$

$\mathbf{Z}/4\mathbf{Z}? \models \forall x\ (x = 0\ \lor\ \exists y\ x \cdot y = 1)?$

$\cdot^{Z/4Z}$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$\mathbf{Z}/4\mathbf{Z}? \models \forall x\ (x = 0\ \vee\ \exists y\ x \cdot y = 1)?$

G moves, chooses $x = 2$

$\cdot_{Z/4Z}$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$\mathbf{Z}/4\mathbf{Z}? \models \forall x\ (x = 0\ \vee\ \exists y\ x \cdot y = 1)?$

G moves, chooses $x = 2$

$\mathbf{Z}/4\mathbf{Z}[2/x]? \models (x = 0\ \vee\ \exists y\ x \cdot y = 1)?$

$\cdot_{Z/4Z}$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$\mathbf{Z}/4\mathbf{Z}? \models \forall x\ (x = 0\ \vee\ \exists y\ x \cdot y = 1)?$

G moves, chooses $x = 2$

$\mathbf{Z}/4\mathbf{Z}[2/x]? \models (x = 0\ \vee\ \exists y\ x \cdot y = 1)?$

D moves, chooses clause $(x = 0)$

$\cdot^{Z/4Z}$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$\mathbf{Z}/4\mathbf{Z}? \models \forall x\ (x = 0\ \vee\ \exists y\ x \cdot y = 1)?$

G moves, chooses $x = 2$

$\mathbf{Z}/4\mathbf{Z}[2/x]? \models (x = 0\ \vee\ \exists y\ x \cdot y = 1)?$

D moves, chooses clause $(x = 0)$

$\mathbf{Z}/4\mathbf{Z}[2/x]? \models x = 0?$

$\cdot^{Z/4Z}$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$\mathbf{Z}/4\mathbf{Z}? \models \forall x\ (x = 0\ \vee\ \exists y\ x \cdot y = 1)?$

G moves, chooses $x = 2$

$\mathbf{Z}/4\mathbf{Z}[2/x]? \models (x = 0\ \vee\ \exists y\ x \cdot y = 1)?$

D moves, chooses clause $(x = 0)$

$\mathbf{Z}/4\mathbf{Z}[2/x]? \models x = 0?$

No, **G** wins!

$\mathbf{Z}/4\mathbf{Z} \not\models \forall x\ (x = 0\ \vee\ \exists y\ x \cdot y = 1)$

$\cdot_{Z/4Z}$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Tarski's Def. of Truth and Truth Game are Equivalent

Thm. For any Σ, $\varphi \in \mathrm{PredCalc}\Sigma$, in NNF, $\mathcal{W} \in \mathrm{World}[\Sigma]$,

$$\textbf{D} \text{ wins the truth game on } \mathcal{W}, \varphi \quad \text{iff} \quad \mathcal{W} \models \varphi$$
$$\textbf{G} \text{ wins the truth game on } \mathcal{W}, \varphi \quad \text{iff} \quad \mathcal{W} \models \sim\varphi$$

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Proof: By induction on the structure of φ.

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base case: φ is a literal. By definition, **D** wins iff $\mathcal{W} \models \varphi$. $\checkmark$

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Proof: By induction on the structure of φ.

base case: φ is a literal. By definition, **D** wins iff $\mathcal{W} \models \varphi$. ✓

inductive case: **G**'s move: $\varphi = \forall x\ (\psi)$ or $\varphi = (\alpha \wedge \beta)$.
Assume $W \models \varphi$. No matter what **G** does, the result remains true. By **indHyp**, **D** wins the remaining game, thus **D** wins the game on W and φ.

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D wins the truth game on $\mathcal{W}, \varphi$ iff $\mathcal{W} \models \varphi$

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Proof: By induction on the structure of φ.

base case: φ is a literal. By definition, **D** wins iff $\mathcal{W} \models \varphi$. ✓

inductive case: **G**'s move: $\varphi = \forall x\ (\psi)$ or $\varphi = (\alpha \wedge \beta)$.
Assume $W \models \varphi$. No matter what **G** does, the result remains true. By **indHyp**, **D** wins the remaining game, thus **D** wins the game on W and φ.
Assume $W \not\models \varphi$. Then **G** has at least one move so that the result remains false. By **indHyp**, **G** wins the remaining game, thus **G** wins the game on W and φ.

Tarski's Def. of Truth and Truth Game are Equivalent

Thm. For any Σ, $\varphi \in \mathrm{PredCalc}\Sigma$, in NNF, $\mathcal{W} \in \mathrm{World}[\Sigma]$,

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Assume $W \models \varphi$. No matter what **G** does, the result remains true. By **indHyp**, **D** wins the remaining game, thus **D** wins the game on W and φ.
Assume $W \not\models \varphi$. Then **G** has at least one move so that the result remains false. By **indHyp**, **G** wins the remaining game, thus **G** wins the game on W and φ.
The case for **D**'s move is similar. If $W \models \varphi$ then **D** has a move that preserves this situation, otherwise, he does not. □