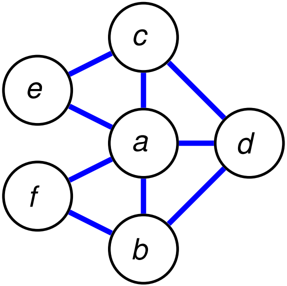
CS250: Discrete Math for Computer Science

L21: Euler's Formula and Well Ordering

$v = |V| \quad e = |E| \quad f = \#$ of faces

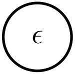
G_1



connected plane graphs

G	v	e	f	$v - e + f$
G_1	6	9	5	2

T_0



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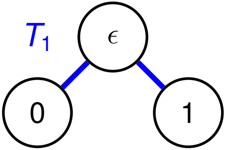

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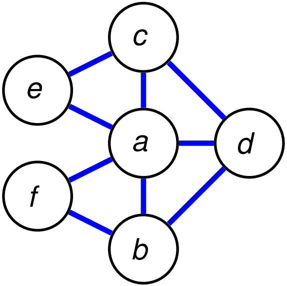

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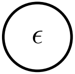
T_0



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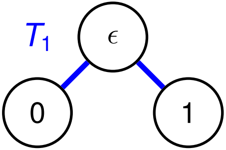

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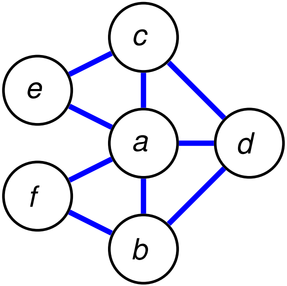

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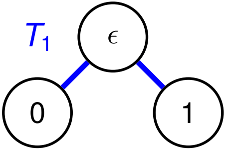
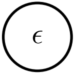


T_0



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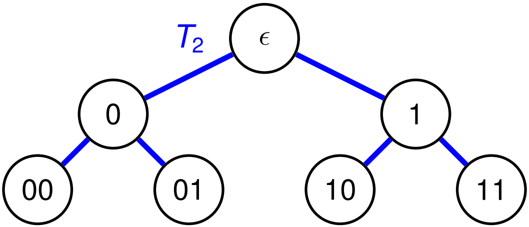

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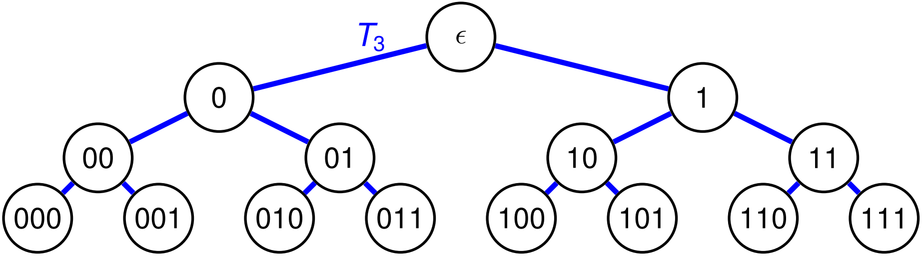
T_3	15	14	1	2



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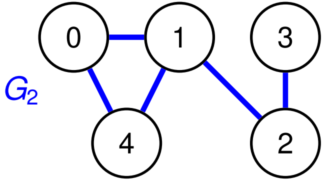

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Proof: Show by **induction** that $\mathbf{Z}^+ \models \forall x\ \alpha(x)$

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base case: $\alpha(1)$: Since $v + e = 1$, $v = 1$ and $e = 0$ and thus $f = 1$. Thus $v - e + f = 2$. ✓

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case 1: G is a tree. Let ℓ be a **leaf** of G, and let (a, ℓ) be the unique edge to a.

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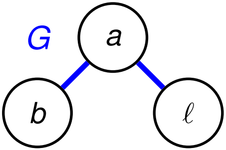

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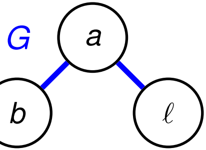

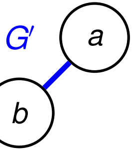

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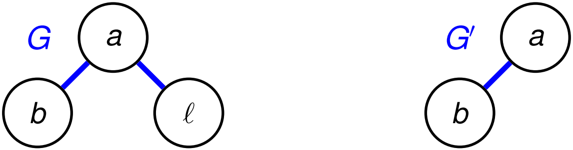

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By **indHyp**, G' satisfies $v'-e'+f'=2$ where $v'=v-1$, $e'=e-1$, $f'=f$. Thus, $v-e+f=2$.

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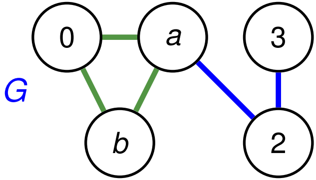

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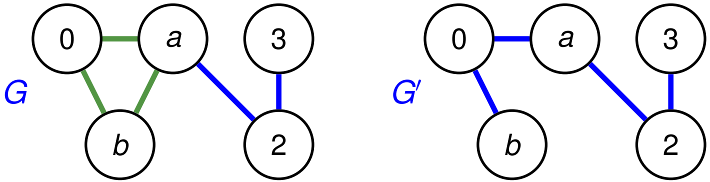

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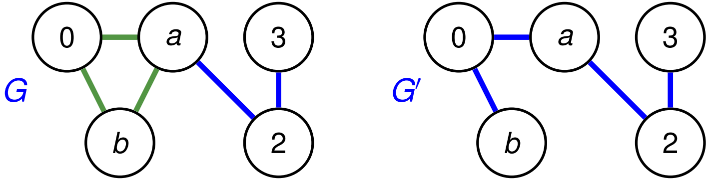

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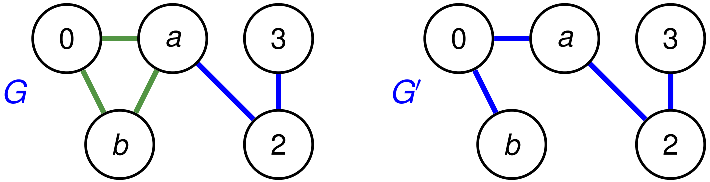

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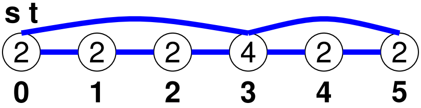

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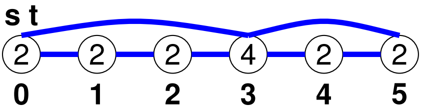

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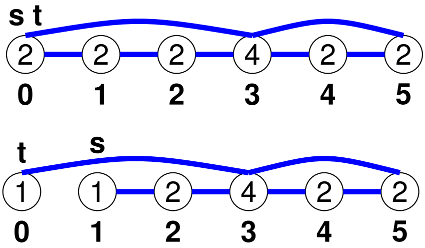

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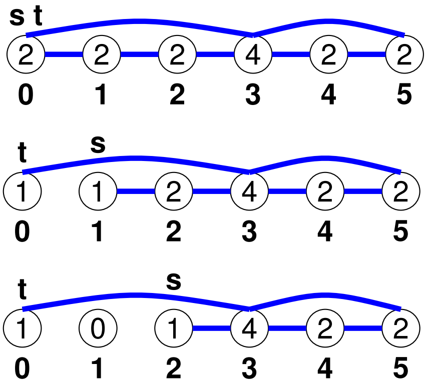

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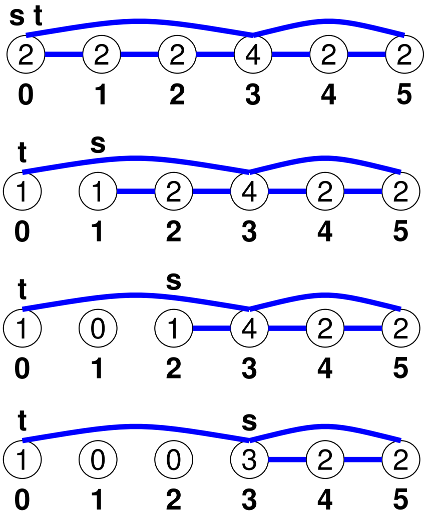

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You must end

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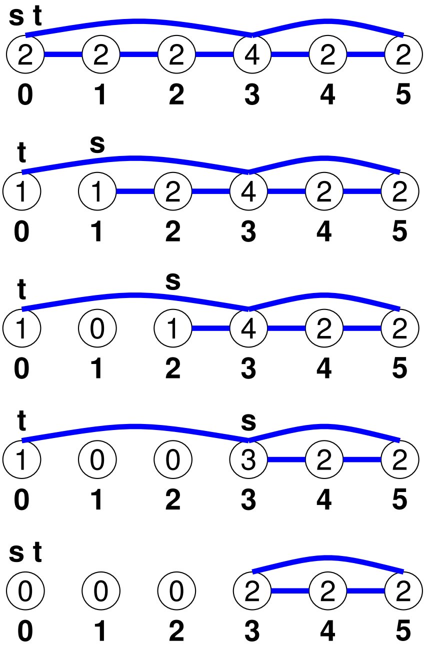

$$
\begin{aligned}
\mathrm{EC}(G) \overset{\text{def}}{=} \ & G \text{ is connected } \wedge \forall v \notin \{s,t\} \ \deg(v) \text{ is even } \wedge \\
& \big(s = t \wedge \deg(s) = \deg(t) \text{ is even } \quad \vee \\
& \ s \neq t \wedge \deg(s), \deg(t) \text{ are odd} \qquad \big)
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Lemma For any undirected graph, G, if $\mathrm{EC}(G)$ then any exhautive bb walk from s must end at t.

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Lemma For any undirected graph, G, if $\mathrm{EC}(G)$ then any exhautive bb walk from s must end at t.

Proof: By induction on $e(G)$. We will prove $\mathbf{N} \models \forall x\ \alpha(x)$.

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base case: If $e(G) = 0 \wedge \mathrm{EC}(G)$, then G is a single vertex with no edges. It's only walk is the empty walk which starts at s and ends at t. ✓

$$\alpha(x) \stackrel{\text{def}}{=} \forall G\,(\mathsf{EC}(G) \wedge e(G) = x \rightarrow \text{any exhaustive bb walk from s ends in t})$$

inductive case: Assume
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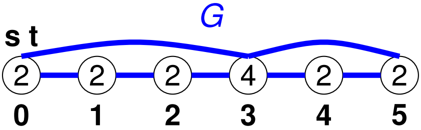

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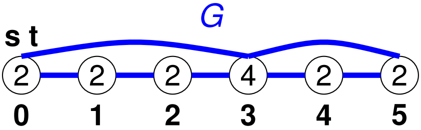

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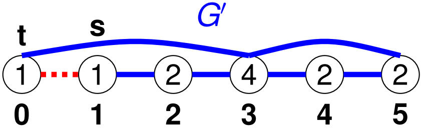

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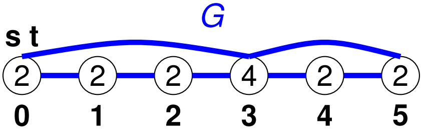

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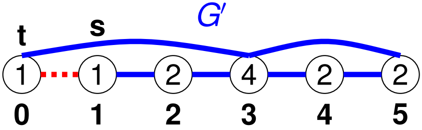

$\text{EC}(G') \wedge e(G) = x_0$

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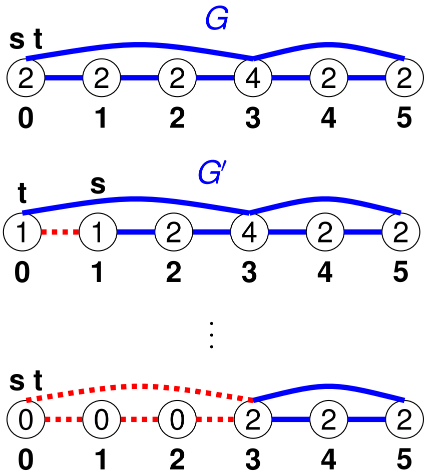
Thus, by **indHyp** the walk from G' must end at t. □



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Thus, $S = \emptyset$, i.e, $\mathbf{N} \models \forall x \; (\alpha(x))$. $\square$