L19: Eulerian Graphs
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Today, all graphs are undirected

![Diagram of undirected graph with vertices 0, 1, and 2 connected in a cycle with blue lines.]
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Is it possible to traverse each of the seven bridges of the city of Königsberg returning to your starting point, without crossing a single bridge twice?
Def. A walk of length $r$ on graph $G$ from $s$ to $t$ is a sequence of $r + 1$ vertices starting at $s$ and ending at $t$ so that each step is along an edge.

$$w = (s = v_0, v_1, v_2, \ldots, v_{r-1}, v_r = t) \quad (v_i, v_{i+1}) \in E^G, 0 \leq i < r$$
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\( G_1 \)

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(0, 3) \ is a walk of length 1 from \( s \) to \( t \)
Def. A path, \( p = (v_0, v_1, \ldots, v_r) \), is a walk with no repeated vertices or edges (except the start and end points may be equal).

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A graph is connected iff it has a path between every pair of vertices. A connected component, \( C \), of an undirected graph, \( G \), is a maximal subgraph of \( G \) that is connected.

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An undirected forest is an acyclic undirected graph. A tree is a connected forest.

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Walks on Graphs

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![Diagram of a graph with nodes 0, 1, 2, 3 and edges between them, illustrating the bb walk from s to t.]
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From $s$, take an

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on $G$, i.e., **until**

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Degree of a Vertex

**Def.** In an undirected graph, the *degree* of a vertex $v$ is the number of neighbors that $v$ has, but loops count as 2:

$$\deg(v) = \left| \{ w \neq v \mid (v, w) \in E^G \} \right| + \begin{cases} 2 & \text{if } v \text{ has a loop} \\ 0 & \text{otherwise} \end{cases}$$

If all the vertices of $G$ have degree $d$, then we say that $G$ is *regular* of degree $d$.

Consider graph $G_1$ and graph $G_2$. Graph $G_1$ is regular of degree 2, but graph $G_2$ is not regular because $\deg(1) \neq \deg(3)$. 
The Parity of the Degrees is Key

**Def.** An **Eulerian walk** in a graph $G$ is a walk from $s$ to $t$ that traverses every edge exactly once and every vertex at least once.

$$\text{EC}(G) \overset{\text{def}}{=} G \text{ is connected } \land \forall v \not\in \{s, t\} \text{ \ deg}(v) \text{ is even } \land \left( s = t \land \text{deg}(s) = \text{deg}(t) \text{ is even} \lor s \neq t \land \text{deg}(s), \text{deg}(t) \text{ are odd} \right)$$
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**Thm.** [Euler] $G$ has an Eulerian walk from $s$ to $t$ iff $\text{EC}(G)$. 
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Claim $G$ has an Eulerian walk $\rightarrow$ EC($G$).
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**Claim** G has an Eulerian walk \( \rightarrow \) EC(G).

**Proof:** For all vertices, \( v \), besides \( s \) and \( t \), the walk must leave \( v \) the same number of times that it enters \( v \). Thus, \( \deg(v) \) is even.
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\textbf{Claim} \ G \text{ has an Eulerian walk } \rightarrow \ EC(G). \\

\textbf{Proof:} \quad \text{For all vertices, } v, \text{ besides } s \text{ and } t, \text{ the walk must leave } v \text{ the same number of times that it enters } v. \text{ Thus, } \text{deg}(v) \text{ is even.} \\
\text{If } s = t, \text{ then } \text{deg}(s) = \text{deg}(t) \text{ is even for the same reason.}
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If \( s \neq t \), then \( s \) is left once more than it is reached and \( t \) is reached once more than it is left, so their degrees are both odd.
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Since an Eulerian walk visits every vertex, \( G \) is connected. \( \square \)
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\textbf{Claim} \ EC(G) \rightarrow G \text{ has an Eulerian walk.} \\

We will prove the Claim by induction: \( \mathbb{N} \models \forall x \ \alpha(x), \) where \\

\[ \alpha(x) \overset{\text{def}}{=} \forall G (|E^G| \leq x \land EC(G) \rightarrow G \text{ has an Eulerian walk}) \]
Claim $EC(G) \rightarrow G$ has an Eulerian walk.

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**base case:** $\alpha(0)$: Let $G$ be an arbitrary graph with 0 edges and $EC(G)$. Since $G$ is connected and has no edges it must consist of a single vertex, $s = t$. Thus, the empty walk is an Eulerian walk from $s$ to $t$. 
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**Next time:** we’ll prove the **inductive case:** \( \alpha(x_0) \rightarrow \alpha(x_0 + 1) \).