CS250: Discrete Math for Computer Science

L18: Mathematical Induction

Induction: a proof rule to prove $\mathbf{N} \models \forall x\ \alpha(x)$

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- **assume** $\alpha(x_0)$, the **inductive hypothesis** (**indHyp**)
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Check: for $x = 0$,

$$\sum_{i=1}^{0} i = 0 = 0 = \frac{0(0+1)}{2} \quad \alpha(0) \checkmark$$

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$$\sum_{i=1}^{2} i = 1 + 2 = 3 = \frac{2(2+1)}{2} \quad \alpha(2) \checkmark$$

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$$\sum_{i=1}^{3} i = 1+2+3 = 6 = \frac{3(3+1)}{2} \quad \alpha(3) \checkmark$$

Prop. $\forall x \sum_{i=1}^{x} i \quad = \quad \frac{x(x+1)}{2}$

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$$\sum_{i=1}^{x_0} i = \frac{x_0(x_0+1)}{2} \qquad \textbf{indHyp}$$

Prop. $\forall x \sum_{i=1}^{x} i = \frac{x(x+1)}{2}$

Proof: Show $\forall x\ \alpha(x)$, by **induction**, $\alpha(x) \stackrel{\text{def}}{=} \left(\sum_{i=1}^{x} i = \frac{x(x+1)}{2}\right)$

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$$\sum_{i=1}^{x_0} i = \frac{x_0(x_0+1)}{2} \qquad \textbf{indHyp}$$

$$\sum_{i=1}^{x_0+1} i = x_0 + 1 + \sum_{i=1}^{x_0} i =$$

Prop. $\forall x \sum_{i=1}^{x} i = \frac{x(x+1)}{2}$

Proof: Show $\forall x\ \alpha(x)$, by **induction**, $\alpha(x) \stackrel{\text{def}}{=} \left(\sum_{i=1}^{x} i = \frac{x(x+1)}{2}\right)$

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inductive case: assume $\alpha(x_0)$ and prove $\alpha(x_0+1)$

$$\sum_{i=1}^{x_0} i = \frac{x_0(x_0+1)}{2} \qquad \textbf{indHyp}$$

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$$= \frac{x_0(x_0+1) + 2(x_0+1)}{2}$$

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$$= \frac{x_0(x_0+1) + 2(x_0+1)}{2} = \frac{(x_0+1)(x_0+2)}{2} \quad ✓$$

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We have proved $\forall x\ \alpha(x)$ by induction. □

iClicker 18.1 For the previous inductive proof of $\forall x\ \alpha(x)$, which of the following formulas is $\alpha(x)$?

A: $$\sum_{i=1}^{x} i = \frac{x(x+1)}{2}$$

B: $$\sum_{i=1}^{x_0} i = \frac{x_0(x_0+1)}{2}$$

C: $$\sum_{i=1}^{0} i = \frac{(0)(0+1)}{2}$$

iClicker 18.2 For the previous inductive proof of $\forall x \; \alpha(x)$, which of the following formulas is the base case?

A: $$\sum_{i=1}^{x} i = \frac{x(x+1)}{2}$$

B: $$\sum_{i=1}^{x_0} i = \frac{x_0(x_0+1)}{2}$$

C: $$\sum_{i=1}^{0} i = \frac{(0)(0+1)}{2}$$

iClicker 18.3 For the previous inductive proof of $\forall x\ \alpha(x)$, which of the following formulas is the inductive hypothesis?

A: $$\sum_{i=1}^{x} i = \frac{x(x+1)}{2}$$

B: $$\sum_{i=1}^{x_0} i = \frac{x_0(x_0+1)}{2}$$

C: $$\sum_{i=1}^{0} i = \frac{(0)(0+1)}{2}$$

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base case: $\alpha(0)$ ✓

Induction: prove $\mathbf{N} \models \forall x\ \alpha(x)$

base case:	$\alpha(0)$	✓
inductive case:		
With x_0 arbitrary, **assume**	$\alpha(x_0)$	**indHyp**

Induction: prove $\mathbf{N} \models \forall x\ \alpha(x)$

base case:	$\alpha(0)$	✓
inductive case:		
With x_0 arbitrary, **assume**	$\alpha(x_0)$	**indHyp**
	$\vdots$	

Induction: prove $\mathbf{N} \models \forall x\ \alpha(x)$

base case:	$\alpha(0)$	✓
inductive case:		
With x_0 arbitrary, **assume**	$\alpha(x_0)$	**indHyp**
	$\vdots$	
Prove	$\alpha(x_0 + 1)$	✓

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base case:	$\alpha(0)$	✓
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With x_0 arbitrary, **assume**	$\alpha(x_0)$	**indHyp**
	$\vdots$	
Prove	$\alpha(x_0 + 1)$	✓
	$\alpha(x_0) \to \alpha(x_0 + 1)$	$\to$-i

Induction: prove $\mathbf{N} \models \forall x\ \alpha(x)$

base case:	$\alpha(0)$	✓
inductive case:		
With x_0 arbitrary, **assume**	$\alpha(x_0)$	**indHyp**
	$\vdots$	
Prove	$\alpha(x_0 + 1)$	✓
	$\alpha(x_0) \to \alpha(x_0 + 1)$	$\to$-i
	$\forall y\ (\alpha(y) \to \alpha(y + 1))$	$\forall$-i

Induction: prove $\mathbf{N} \models \forall x\ \alpha(x)$

base case:	$\alpha(0)$	✓
inductive case:		
With x_0 arbitrary, **assume**	$\alpha(x_0)$	**indHyp**
	$\vdots$	
Prove	$\alpha(x_0 + 1)$	✓
	$\alpha(x_0) \rightarrow \alpha(x_0 + 1)$	$\rightarrow$-i
	$\forall y\ (\alpha(y) \rightarrow \alpha(y + 1))$	$\forall$-i

$$\underbrace{\alpha(0)} \quad \wedge \quad \underbrace{\forall y\ (\alpha(y) \rightarrow \alpha(y + 1))} \quad \rightarrow \quad \forall x\ \alpha(x)$$

Induction: prove $\mathbf{N} \models \forall x\ \alpha(x)$

base case:	$\alpha(0)$	✓
inductive case:		
With x_0 arbitrary, **assume**	$\alpha(x_0)$	**indHyp**
	$\vdots$	
Prove	$\alpha(x_0+1)$	✓
	$\alpha(x_0) \rightarrow \alpha(x_0+1)$	$\rightarrow$-i
	$\forall y\ (\alpha(y) \rightarrow \alpha(y+1))$	$\forall$-i

$$\underbrace{\alpha(0)}_{\textbf{base case}} \quad \wedge \quad \underbrace{\forall y\ (\alpha(y) \rightarrow \alpha(y+1))}_{} \quad \rightarrow \quad \forall x\ \alpha(x)$$

Induction: prove $\mathbf{N} \models \forall x\ \alpha(x)$

base case: $\alpha(0)$ ✓

inductive case:

With x_0 arbitrary, **assume** $\alpha(x_0)$ **indHyp**

$$\vdots$$

Prove $\alpha(x_0 + 1)$ ✓

$$\alpha(x_0) \rightarrow \alpha(x_0 + 1) \qquad \rightarrow\text{-i}$$

$$\forall y\ (\alpha(y) \rightarrow \alpha(y + 1)) \qquad \forall\text{-i}$$

$$\underbrace{\alpha(0)}_{\textbf{base case}} \quad \wedge \quad \underbrace{\forall y\ (\alpha(y) \rightarrow \alpha(y+1))}_{\textbf{inductive case}} \quad \rightarrow \quad \forall x\ \alpha(x)$$

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Check: for $x = 0,\ 1,$

$$\sum_{i=1}^{0} (2i\text{-}1) \quad = \quad 0 \quad = \quad 0 \quad = \quad 0^2 \quad \alpha(0) \checkmark$$

$$\sum_{i=1}^{1} (2i\text{-}1) \quad = \quad 1 \quad = \quad 1 \quad = \quad 1^2 \quad \alpha(1) \checkmark$$

Show: $\forall x\ (\alpha(x))$ $\alpha(x) \stackrel{\text{def}}{=} \forall x \sum_{i=1}^{x}(2i-1) = x^2$

Check: for $x = 0,\ 1,\ 2,$

$$\sum_{i=1}^{0} (2i\text{-}1) = 0 = 0 = 0^2 \quad \alpha(0) \checkmark$$

$$\sum_{i=1}^{1} (2i\text{-}1) = 1 = 1 = 1^2 \quad \alpha(1) \checkmark$$

$$\sum_{i=1}^{2} (2i\text{-}1) = 1 + 3 = 4 = 2^2 \quad \alpha(2) \checkmark$$

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Check: for $x = 0,\ 1,\ 2,\ 3.$

$$\sum_{i=1}^{0}(2i-1) = 0 = 0 = 0^2 \quad \alpha(0)\ \checkmark$$

$$\sum_{i=1}^{1}(2i-1) = 1 = 1 = 1^2 \quad \alpha(1)\ \checkmark$$

$$\sum_{i=1}^{2}(2i-1) = 1+3 = 4 = 2^2 \quad \alpha(2)\ \checkmark$$

$$\sum_{i=1}^{3}(2i-1) = 1+3+5 = 9 = 3^2 \quad \alpha(3)\ \checkmark$$

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Proof: Show $\forall x\ \alpha(x)$ by induction, $\alpha(x) \stackrel{\text{def}}{=} \left(\sum_{i=1}^{x}(2i-1) = x^2\right)$

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Proof: Show $\forall x\ \alpha(x)$ by induction, $\quad \alpha(x) \stackrel{\text{def}}{=} \left(\sum_{i=1}^{x}(2i-1) = x^2\right)$

base case: $\alpha(0)$: $\quad \sum_{i=1}^{0}(2i-1) \;=\; 0 \;=\; 0^2$

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base case: $\alpha(0)$: $\quad \sum_{i=1}^{0}(2i-1) \ = \ 0 \ = \ 0^2$

inductive case: assume $\alpha(x_0)$: $\quad \sum_{i=1}^{x_0}(2i-1) \quad = \quad x_0^2$

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$$\sum_{i=1}^{x_0+1}(2i-1) \quad = \quad \left(\sum_{i=1}^{x_0}(2i-1)\right) \ + \ 2(x_0+1)-1$$

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$$\begin{aligned} \sum_{i=1}^{x_0+1}(2i-1) \quad &= \quad \left(\sum_{i=1}^{x_0}(2i-1)\right) \ + \ 2(x_0+1)-1 \\ &= \quad x_0^2 + 2x_0 + 1 \qquad \textbf{indHyp} \end{aligned}$$

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$$
\begin{aligned}
\sum_{i=1}^{x_0+1}(2i-1) \quad &= \quad \left(\sum_{i=1}^{x_0}(2i-1)\right) \ + \ 2(x_0+1)-1 \\
&= \quad x_0^2 + 2x_0 + 1 && \textbf{indHyp} \\
&= \quad (x_0+1)^2 && \square
\end{aligned}
$$

Show: $\forall x\ (\alpha(x))$ $\qquad$ $\alpha(x) \stackrel{\text{def}}{=} \sum_{i=0}^{x} 2^{-i} = 2 - 2^{-x}$

Show: $\forall x\ (\alpha(x))$ $\qquad$ $\alpha(x) \stackrel{\text{def}}{=} \sum_{i=0}^{x} 2^{-i} = 2 - 2^{-x}$

Check: for $x = 0$,

$$\sum_{i=0}^{0} 2^{-i} \quad = \quad 1 \quad = \quad 1 \quad = \quad 2 - 2^{-0} \qquad \alpha(0) \checkmark$$

Show: $\forall x\ (\alpha(x))$ $\qquad \alpha(x) \stackrel{\text{def}}{=} \sum_{i=0}^{x} 2^{-i} = 2 - 2^{-x}$

Check: for $x = 0,\ 1,$

$$\sum_{i=0}^{0} 2^{-i} = 1 = 1 = 2 - 2^{-0} \qquad \alpha(0) \checkmark$$

$$\sum_{i=0}^{1} 2^{-i} = 1 + \frac{1}{2} = \frac{3}{2} = 2 - 2^{-1} \qquad \alpha(1) \checkmark$$

Show: $\forall x\ (\alpha(x))$ $\qquad \alpha(x) \stackrel{\text{def}}{=} \sum_{i=0}^{x} 2^{-i} = 2 - 2^{-x}$

Check: for $x = 0,\ 1,\ 2,$

$$\sum_{i=0}^{0} 2^{-i} = 1 = 1 = 2 - 2^{-0} \quad \alpha(0) \checkmark$$

$$\sum_{i=0}^{1} 2^{-i} = 1 + \frac{1}{2} = \frac{3}{2} = 2 - 2^{-1} \quad \alpha(1) \checkmark$$

$$\sum_{i=0}^{2} 2^{-i} = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} = 2 - 2^{-2} \quad \alpha(2) \checkmark$$

Show: $\forall x\,(\alpha(x))$ $\qquad \alpha(x) \stackrel{\text{def}}{=} \sum_{i=0}^{x} 2^{-i} = 2 - 2^{-x}$

Check: for $x = 0,\ 1,\ 2,\ 3.$

$$\sum_{i=0}^{0} 2^{-i} = 1 = 1 = 2 - 2^{-0} \qquad \alpha(0) \checkmark$$

$$\sum_{i=0}^{1} 2^{-i} = 1 + \frac{1}{2} = \frac{3}{2} = 2 - 2^{-1} \qquad \alpha(1) \checkmark$$

$$\sum_{i=0}^{2} 2^{-i} = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} = 2 - 2^{-2} \qquad \alpha(2) \checkmark$$

$$\sum_{i=0}^{3} 2^{-i} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8} = 2 - 2^{-3} \qquad \alpha(3) \checkmark$$

Prop. $\forall x \quad \sum_{i=0}^{x} 2^{-i} \quad = \quad 2 - 2^{-x}$

Prop. $\forall x \quad \sum_{i=0}^{x} 2^{-i} \quad = \quad 2 - 2^{-x}$

Proof: $\forall x \ \alpha(x)$, by **induction**, $\alpha(x) \stackrel{\text{def}}{=} \left(\sum_{i=1}^{x} 2^{-i} = 2 - 2^{-x}\right)$

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base case: $\alpha(0)$: $\quad \sum_{i=0}^{0}(2^{-i}) \ = \ 2^0 \ = \ 1 \ = \ 2 - 2^0 \quad \checkmark$

Prop. $\forall x \quad \sum_{i=0}^{x} 2^{-i} \quad = \quad 2 - 2^{-x}$

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inductive case: $\alpha(x_0)$: $\quad \sum_{i=0}^{x_0} 2^{-i} \quad = \quad 2 - 2^{-x_0}$

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base case: $\alpha(0)$: $\quad \sum_{i=0}^{0}(2^{-i}) \;=\; 2^0 \;=\; 1 \;=\; 2 - 2^0 \quad \checkmark$

inductive case: $\alpha(x_0)$: $\quad \sum_{i=0}^{x_0} 2^{-i} \quad = \quad 2 - 2^{-x_0}$

$$\sum_{i=0}^{x_0+1} 2^{-i} \quad = \quad \left(\sum_{i=0}^{x_0} 2^{-i}\right) + 2^{-(x_0+1)}$$

Prop. $\forall x \quad \sum_{i=0}^{x} 2^{-i} \quad = \quad 2 - 2^{-x}$

Proof: $\forall x\ \alpha(x)$, by **induction**, $\alpha(x) \stackrel{\text{def}}{=} \left(\sum_{i=1}^{x} 2^{-i} = 2 - 2^{-x}\right)$

base case: $\alpha(0)$: $\quad \sum_{i=0}^{0}(2^{-i}) \ = \ 2^0 \ = \ 1 \ = \ 2 - 2^0 \quad \checkmark$

inductive case: $\alpha(x_0)$: $\quad \sum_{i=0}^{x_0} 2^{-i} \quad = \quad 2 - 2^{-x_0}$

$$\begin{aligned}
\sum_{i=0}^{x_0+1} 2^{-i} \quad &= \quad \left(\sum_{i=0}^{x_0} 2^{-i}\right) + 2^{-(x_0+1)} \\
&= \quad \left(2 - 2^{-x_0}\right) + 2^{-(x_0+1)} \qquad \textbf{indHyp}
\end{aligned}$$

Prop. $\quad \forall x \quad \sum_{i=0}^{x} 2^{-i} \quad = \quad 2 - 2^{-x}$

Proof: $\forall x\ \alpha(x)$, by **induction**, $\alpha(x) \stackrel{\text{def}}{=} \left(\sum_{i=1}^{x} 2^{-i} = 2 - 2^{-x}\right)$

base case: $\alpha(0)$: $\quad \sum_{i=0}^{0}(2^{-i}) \; = \; 2^0 \; = \; 1 \; = \; 2 - 2^0 \quad \checkmark$

inductive case: $\alpha(x_0)$: $\quad \sum_{i=0}^{x_0} 2^{-i} \quad = \quad 2 - 2^{-x_0}$

$$
\begin{aligned}
\sum_{i=0}^{x_0+1} 2^{-i} \quad &= \quad \left(\sum_{i=0}^{x_0} 2^{-i}\right) + 2^{-(x_0+1)} \\
&= \quad \left(2 - 2^{-x_0}\right) + 2^{-(x_0+1)} \qquad \textbf{indHyp} \\
&= \quad 2 - 2^{-(x_0+1)} \qquad\qquad \checkmark \qquad \square
\end{aligned}
$$

The Induction Rule

$$\frac{\alpha(0) \quad \forall y\,(\alpha(y) \;\rightarrow\; \alpha(y+1))}{\forall x\,(\alpha(x))}$$

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base case: $\alpha(0)$ ✓

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$\vdots$

The Induction Rule

$$\frac{\alpha(0) \quad \forall y\,(\alpha(y) \;\rightarrow\; \alpha(y+1))}{\forall x\,(\alpha(x))}$$

base case: $\alpha(0)$ ✓

inductive case: **assume indHyp:** $\alpha(x_0)$

$$\vdots$$

$\alpha(x_0 + 1)$ ✓

The Induction Rule

$$\frac{\alpha(0) \quad \forall y\,(\alpha(y) \;\rightarrow\; \alpha(y+1))}{\forall x\,(\alpha(x))}$$

base case: $\alpha(0)$ ✓

inductive case: **assume indHyp:** $\alpha(x_0)$

$$\vdots$$

$\alpha(x_0 + 1)$ ✓

$\alpha(x_0) \rightarrow \alpha(x_0 + 1)$ $\rightarrow$-i

The Induction Rule

$$\frac{\alpha(0) \quad \forall y\,(\alpha(y) \;\rightarrow\; \alpha(y+1))}{\forall x\,(\alpha(x))}$$

base case:		$\alpha(0)$	✓
inductive case:	**assume indHyp:**	$\alpha(x_0)$	
		$\vdots$	
		$\alpha(x_0+1)$	✓
		$\alpha(x_0) \rightarrow \alpha(x_0+1)$	$\rightarrow$-i
		$\forall y\,(\alpha(y) \rightarrow \alpha(y+1))$	$\forall$-i

The Induction Rule

$$\frac{\alpha(0) \quad \forall y\,(\alpha(y) \;\rightarrow\; \alpha(y+1))}{\forall x\,(\alpha(x))}$$

base case:		$\alpha(0)$	✓
inductive case:	**assume indHyp:**	$\alpha(x_0)$	
		$\vdots$	
		$\alpha(x_0+1)$	✓
		$\alpha(x_0) \rightarrow \alpha(x_0+1)$	$\rightarrow$-i
		$\forall y\,(\alpha(y) \rightarrow \alpha(y+1))$	$\forall$-i
		$\forall x\,(\alpha(x))$	**Induction**