L12: Last Two Natural Deduction Rules: ∀-i and ∃-e
Tarski’s Definition of Truth

$G \models t_1 = t_2$ \iff $t_1^G = t_2^G$

$G \models P(t_1, \ldots, t_a)$ \iff $(t_1^G, \ldots, t_a^G) \in P^G$

$G \models \neg \alpha$ \iff $G \not\models \alpha$

$G \models \alpha \land \beta$ \iff $G \models \alpha$ and $G \models \beta$

$G \models \alpha \lor \beta$ \iff $G \models \alpha$ or $G \models \beta$

$G \models \forall x(\alpha)$ \iff for all $a \in |G|$ \quad $G[a/x] \models \alpha$

$G \models \exists x(\alpha)$ \iff exists $a \in |G|$ \quad $G[a/x] \models \alpha$
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PredCalc Natural Deduction Rules

**Proviso** for ∀-i and ∃-e: \( x_0 \) is a “new” variable, i.e., it does not appear in \( \varphi, \psi, \) or \( \Gamma, \) i.e., in any visible assumption.

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<tr>
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<td>( \exists )</td>
<td>( \varphi[t/x] )</td>
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1 \[\sim \alpha \lor \sim \beta\]
2 \[\alpha \land \beta\]
3 \[\sim \alpha\]
4 \[\alpha\] \land-e, 2
5 \[\textbf{F}\] \textbf{F-i}, 3, 4
6 \[\sim \beta\]
7 \[\beta\] \land-e, 2
8 \[\textbf{F}\] \textbf{F-i}, 6, 7
9 \[\textbf{F}\] \lor-e, 1, 3–5, 6–8
10 \[\sim (\alpha \land \beta)\] \textbf{F-e}, 2–9
From **assumption** \((\sim \alpha \lor \sim \beta)\), we **proved** \(\sim (\alpha \land \beta)\).
From assumption \((\sim \alpha \lor \sim \beta)\), we proved \((\sim (\alpha \land \beta))\).

\((\sim \alpha \lor \sim \beta) \vdash \sim (\alpha \land \beta)\)

“\(\vdash\)” is read “proves”.

\(~\alpha \lor \sim \beta\)
From **assumption** \((\neg \alpha \lor \neg \beta)\), **we proved** \(\neg (\alpha \land \beta)\).

\((\neg \alpha \lor \neg \beta) \vdash \neg (\alpha \land \beta)\) “\(\vdash\)” is read “**proves**”.

\(\Gamma \vdash \varphi: \quad \Gamma\) **proves** \(\varphi\)

There is a **proof** of \(\varphi\), that **may use assumptions** from the set \(\Gamma\).
From **assumption** \((\sim \alpha \lor \sim \beta)\), we **proved** \(\sim (\alpha \land \beta)\).

\((\sim \alpha \lor \sim \beta) \vdash \sim (\alpha \land \beta)\) \(\vdash\) is read **"proves"**.

\(\Gamma \vdash \varphi: \quad \Gamma \text{ proves } \varphi\)

There is a **proof** of \(\varphi\), that **may use assumptions** from the set \(\Gamma\).

\(\vdash \varphi: \quad \varphi \text{ is a } \textbf{theorem} \text{ of PredCalc.}\)

There is a **proof** of \(\varphi\), with no assumptions.

\[
\begin{align*}
\frac{\varphi \vdash \psi}{\vdash \varphi \rightarrow \psi} \quad \rightarrow\text{-i}
\end{align*}
\]
From **assumption** \((\sim \alpha \lor \sim \beta)\), we **proved** \(\sim (\alpha \land \beta)\).

\[(\sim \alpha \lor \sim \beta) \vdash \sim (\alpha \land \beta)\]  

“\(\vdash\)” is read “**proves**”.

\[\Gamma \vdash \varphi: \quad \Gamma \text{ proves } \varphi\]

There is a **proof** of \(\varphi\), that **may use assumptions** from the set \(\Gamma\).

\[\vdash \varphi: \quad \varphi \text{ is a **theorem** of PredCalc.}\]

There is a **proof** of \(\varphi\), with no assumptions.

\[
\frac{\varphi \vdash \psi}{\vdash \varphi \rightarrow \psi} \quad \rightarrow\text{-i} \quad \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi} \quad \rightarrow\text{-i}
\]
From **assumption** \( \sim \alpha \lor \sim \beta \), we **proved** \( \sim (\alpha \land \beta) \).

\[
(\sim \alpha \lor \sim \beta) \vdash \sim (\alpha \land \beta)
\]

“\( \vdash \)” is read “**proves**”.

\[ \Gamma \vdash \varphi : \text{\( \Gamma \) proves \( \varphi \)} \]

There is a **proof** of \( \varphi \), that **may use assumptions** from the set \( \Gamma \).

\[ \vdash \varphi : \text{\( \varphi \) is a **theorem** of PredCalc.} \]

There is a **proof** of \( \varphi \), with no assumptions.

\[
\frac{\varphi \vdash \psi}{\vdash \varphi \rightarrow \psi} \quad \rightarrow\text{-i} \quad \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi} \quad \rightarrow\text{-i}
\]

\[ \vdash \sim \alpha \lor \sim \beta \rightarrow \sim (\alpha \land \beta) \]
∀-e

\[
\frac{\forall x (\varphi)}{\varphi [t/x]}
\]
Natural Deduction: the Obvious Quantifier Rules

∀-e \quad \frac{\forall x(\varphi)}{\varphi[t/x]} \quad \text{If } \forall x(\varphi) \text{ then for any term } t, \varphi[t/x]

If } \forall x(\varphi) then for any term } t, } \varphi[t/x]
Natural Deduction: the Obvious Quantifier Rules

\[\forall - e \quad \frac{\forall x(\varphi)}{\varphi[t/x]} \quad \text{If } \forall x(\varphi) \text{ then for any term } t, \varphi[t/x]\]

\[\exists - i \quad \frac{\varphi[t/x]}{\exists x(\varphi)}\]
Natural Deduction: the Obvious Quantifier Rules

\[\forall \text{-} e \quad \frac{\forall x(\varphi)}{\varphi[t/x]} \quad \text{If } \forall x(\varphi) \text{ then for any term } t, \varphi[t/x]\]

\[\exists \text{-} i \quad \frac{\varphi[t/x]}{\exists x(\varphi)} \quad \text{If I proved } \varphi[t/x] \text{ then I know } \exists x(\varphi)\]
∀-i

\[ \forall x \left( \varphi \right) \]

**Proviso:** \( x_0 \) does not occur in \( \varphi \), nor in any current assumption.

Informally, we say, “Let \( x_0 \) be arbitrary.” Then we prove \( \varphi[x_0/x] \).

We conclude, “Since \( x_0 \) was arbitrary, it follows that \( \forall x(\varphi) \).”
∀-i  \[ \frac{\varphi[x_0/x]}{\forall x(\varphi)} \]

**Proviso:**

\( x_0 \) does not occur in \( \varphi \),

nor in any current assumption
∀-i

∀-i \quad \frac{\varphi[x_0/x]}{\forall x(\varphi)}

Proviso:
\(x_0\) does not occur in \(\varphi\),
nor in any current assumption

Informally, we say, “Let \(x_0\) be arbitrary.”
∀-i  \[ \frac{\varphi[x_0/x]}{\forall x(\varphi)} \]

Proviso:
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Then we prove \( \varphi[x_0/x] \).
Informally, we say, “Let $x_0$ be arbitrary.”

Then we prove $\varphi[x_0/x]$.

We conclude, “Since $x_0$ was arbitrary, it follows that $\forall x(\varphi)$.”
Why the **proviso** is **necessary**
Why the **proviso** is necessary

\[ \varphi \equiv (x = x_0 \lor x = s) \]
Why the **proviso** is necessary

\[ \varphi \equiv (x = x_0 \lor x = s) \]

\[ \varphi[x_0/x] \equiv (x_0 = x_0 \lor x_0 = s) \]
Why the proviso is necessary

\[ \varphi \equiv (x = x_0 \lor x = s) \]
\[ \varphi[x_0/x] \equiv (x_0 = x_0 \lor x_0 = s) \]

\[ 1 \mid x_0 = x_0 \quad = -i \]
Why the **proviso** is necessary

\[ \varphi \equiv (x = x_0 \lor x = s) \]

\[ \varphi[x_0/x] \equiv (x_0 = x_0 \lor x_0 = s) \]

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<td>2</td>
<td>( x_0 = x_0 \lor x_0 = s )</td>
<td>( \lor-i, 1 )</td>
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Line 3 does not follow because \( x_0 \) occurs in \( \varphi \).
Why the **proviso** is necessary

\[ \varphi \equiv (x = x_0 \lor x = s) \]
\[ \varphi[x_0/x] \equiv (x_0 = x_0 \lor x_0 = s) \]

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1. \( x_0 = x_0 \) \quad =-i
2. \( x_0 = x_0 \lor x_0 = s \) \quad \lor-i, 1
3. \( \forall x (x = x_0 \lor x = s) \) \quad \forall-i, 2

Line 3 does not follow because \( x_0 \) occurs in \( \varphi \).
Why the proviso is necessary

$$\varphi \equiv (x = x_0 \lor x = s)$$

$$\varphi[x_0/x] \equiv (x_0 = x_0 \lor x_0 = s)$$

| 1 | \( x_0 = x_0 \) | \(-i\) \\
| 2 | \( x_0 = x_0 \lor x_0 = s \) | \(\lor\)-i, 1 \\
| 3 | \( \forall x(x = x_0 \lor x = s) \) | \(\forall\)-i, 2 X X X

Line 3 does not follow because \( x_0 \) occurs in \( \varphi \).
\[ \exists \mathbf{x} (\varphi), \varphi[^{x_0}/x] \vdash \psi \]
Informally, we say, "Since we know that something satisfies $\varphi$, we may assume $\varphi[x_0/x]$ for a new variable $x_0$. If we prove $\psi$ using this assumption and $x_0$ does not occur in $\varphi$ nor in $\psi$, nor in any current assumption, then we have proved $\psi$, with no assumptions required."
Informally, we say, “Since we know that something satisfies $\varphi$,“
Informally, we say, “Since we know that something satisfies \( \varphi \),” we may assume \( \varphi[x_0/x] \) for a new variable \( x_0 \).
\( \exists \text{-e} \quad \frac{\exists x(\varphi), \varphi[x_0/x] \vdash \psi}{\psi} \)

Proviso: 
\( x_0 \) does not occur in \( \varphi \) nor in \( \psi \), nor in any current assumption.

Informally, we say, “Since we know that something satisfies \( \varphi \),” we may assume \( \varphi[x_0/x] \) for a new variable \( x_0 \).
If we prove \( \psi \) using this assumption
\exists-e \quad \frac{\exists x(\varphi), \varphi[x_0/x] \vdash \psi}{\psi}

Proviso:  
\(x_0\) does not occur in \(\varphi\) nor in \(\psi\), nor in any current assumption

Informally, we say, “Since we know that something satisfies \(\varphi\),” we may assume \(\varphi[x_0/x]\) for a new variable \(x_0\).  
If we prove \(\psi\) using this assumption 
and \(x_0\) does not occur in \(\psi\)
\[ \exists-e \quad \exists x(\varphi), \varphi[x_0/x] \vdash \psi \]

Proviso:
\( x_0 \) does not occur in \( \varphi \) nor in \( \psi \), nor in any current assumption.

Informally, we say, “Since we know that something satisfies \( \varphi \),” we may assume \( \varphi[x_0/x] \) for a new variable \( x_0 \).

If we prove \( \psi \) using this assumption and \( x_0 \) does not occur in \( \psi \) then we have proved \( \psi \), with no assumptions required.
1. \( x = y \)

2. \( x = z \)
   \[ \underline{x = z} \]

3. \( y = z \)

4. \( x = z \rightarrow y = z \)  \[-i, 2-3\]

5. \( \forall z (x = z \rightarrow y = z) \)
   \[-i, 4\]

6. \( x = x \rightarrow y = x \)
   \[-e, 5\]

7. \( x = x \)
   \[-i\]

8. \( y = x \)
   \[-e, 6, 7\]

9. \( x = y \rightarrow y = x \)
   \[-i, 1-8\]

10. \( \forall xy (x = y \rightarrow y = x) \)
    \[-i, 9\]
Is this “proof” of $\vdash \exists x \forall y E(x, y) \rightarrow \forall y \exists x E(x, y)$ correct?

1. $\exists x \forall y E(x, y)$

2. $\forall y E(x_0, y)$

3. $E(x_0, y_0)$ \hspace{1cm} \forall-e, 2

4. $\exists x E(x, y_0)$ \hspace{1cm} \exists-i, 3

5. $\exists x E(x, y_0)$ \hspace{1cm} \exists-e, 1, 2–4

6. $\forall y \exists x E(x, y)$ \hspace{1cm} \forall-i, 5

7. $\exists x \forall y E(x, y) \rightarrow \forall y \exists x E(x, y)$ \hspace{1cm} \rightarrow-i, 1–6
Is this “proof” of $\vdash \exists x \forall y E(x, y) \rightarrow \forall y \exists x E(x, y)$ correct? Yes.

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<tr>
<th>1</th>
<th>$\exists x \forall y E(x, y)$</th>
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<tbody>
<tr>
<td>2</td>
<td>$\forall y E(x_0, y)$</td>
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<tr>
<td>3</td>
<td>$E(x_0, y_0)$</td>
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<tr>
<td>4</td>
<td>$\exists x E(x, y_0)$</td>
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<tr>
<td>5</td>
<td>$\exists x E(x, y_0)$</td>
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<tr>
<td>6</td>
<td>$\forall y \exists x E(x, y)$</td>
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<tr>
<td>7</td>
<td>$\exists x \forall y E(x, y) \rightarrow \forall y \exists x E(x, y)$</td>
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Is this “proof” of $\vdash \forall y \exists x E(x, y) \rightarrow \exists x \forall y E(x, y)$ correct?

1. $\forall y \exists x E(x, y)$
2. $\exists x E(x, y_0)$  \hspace{1cm} $\forall$-e, 1
3. $E(x_0, y_0)$  \hspace{1cm} \hspace{1cm} 2
4. $\forall y E(x_0, y)$  \hspace{1cm} \hspace{1cm} 3
5. $\exists x \forall y E(x, y)$  \hspace{1cm} \hspace{1cm} 4
6. $\forall y \exists x E(x, y) \rightarrow \exists x \forall y E(x, y)$  \hspace{1cm} \hspace{1cm} 1–5
Is this “proof” of \( \vdash \forall y \exists x E(x, y) \rightarrow \exists x \forall y E(x, y) \) correct?

No. Line 3 does not follow from lines 1 and 2.

\[ G_2 \models \text{lines 1 and 2} \]

\[ G_2 \models \sim E(x_0, y_0) \]
1. Convert to NNF

a. A4  \( \sim \forall x \exists y (A(x) \rightarrow E(x, y)) \)

b. A6  \( \sim \exists x \exists y (E(x, y) \land \sim (A(x) \land R(y)))) \)

c. A5  \( \sim \forall x \exists y (R(x) \rightarrow E(y, x)) \)

d. A3  \( \sim \exists x \exists y \exists z (E(x, y) \land E(y, z) \land y \neq z) \)

Possible Answers:

A1. \( \forall x \exists y (R(x) \land \sim E(y, x)) \)

A2. \( \forall xyz (\sim E(x, y) \lor \sim E(y, z) \rightarrow y = z) \)

A3. \( \forall xyz (\sim E(x, y) \lor \sim E(y, z) \lor y = z) \)

A4. \( \exists x \forall y (A(x) \land \sim E(x, y)) \)

A5. \( \exists x \forall y (R(x) \land \sim E(y, x)) \)

A6. \( \forall x \forall y (\sim E(x, y) \lor (A(x) \land R(y))) \)
2. Match English statements with PredCalc formulas.

a. B7  \( s \) is an A vertex but not an R vertex.

b. B3  Every vertex is either A or R, but not both.

c. B1  Every edge goes from an A vertex to an R vertex.

d. B2  Every R vertex has an incoming edge from an A vertex.

e. B4  Some R vertex has an incoming edge from every A vertex.

Possible Answers:

B1. \( \forall x \forall y (E(x,y) \rightarrow A(x) \land R(y)) \)

B2. \( \forall y \exists x (R(y) \rightarrow A(x) \land E(x,y)) \)

B3. \( \forall x (A(x) \oplus R(x)) \)

B4. \( \exists y \forall x (R(y) \land (A(x) \rightarrow E(x,y)) \)

B5. \( \forall x ((A(x) \lor R(x)) \land (A(x) \leftrightarrow R(x))) \)

B6. \( A(s) \rightarrow \sim R(s) \)

B7. \( A(s) \land \sim R(s) \)

B8. \( \exists y \forall x (R(y) \rightarrow A(x) \rightarrow E(x,y)) \)
3.1 Is this “proof” of \( \vdash R(x) \land A(y) \rightarrow A(y) \land R(x) \) correct?

\[
\begin{array}{c|c}
1 & R(x) \land A(y) \\
2 & R(x) & \land \text{-e, 1} \\
3 & A(y) & \land \text{-e, 1} \\
4 & A(y) \land R(x) & \land \text{-i, 2, 3} \\
5 & R(x) \land A(y) \rightarrow A(y) \land R(x) & \rightarrow \text{-i, 1–4} \\
\end{array}
\]

This proof is correct.
3.2 Is this “proof” of \((\forall x . R(x))(A(x)) \vdash (\exists x . R(x))(A(x))\) correct?

1. \(\forall x (R(x) \rightarrow A(x))\)
2. \(R(x) \rightarrow A(x)\) \(\forall\)-e, 1
3. \(R(x)\)
4. \(A(x)\) \(\rightarrow\)-e, 2, 3
5. \(R(x) \land A(x)\) \(\land\)-i, 3, 4
6. \(A(x)\)
7. \(R(x) \land A(x)\) \(\times\), 3, 6
8. \(R(x) \land A(x)\) \(\times\), 2, 3–5, 6–7
9. \(\exists x (R(x) \land A(x))\) \(\exists\)-i, 8

No, \(G_7 \models\) lines 1 and 6, but not line 7 or 8.

a. \( E2 \quad G_4 \quad \exists x \forall y (E(x, y) \lor E(y, x) \rightarrow x = y) \)

b. \( E3 \quad G_2 \quad \exists y \forall x (A(y) \land \sim E(x, y)) \)

c. \( E4 \quad G_3 \quad \forall xy (E(x, y) \rightarrow E(y, x)) \)

d. \( E1 \quad G_1, G_2 \quad \exists! x \exists! y E(x, y) \)

E1. Exactly one vertex has outdegree exactly 1.

E2. Some vertex is isolated, i.e., has no edge in or out to a different vertex.

E3. Some A vertex has no incoming edge.

E4. The graph is undirected.