CS250: Discrete Math for Computer Science

L12: Last Two Natural Deduction Rules: $\forall$-i and $\exists$-e

Tarski's Definition of Truth

$$
\begin{array}{llll}
G & \models & t_1 = t_2 & \text{iff} \quad t_1^G = t_2^G \\
G & \models & P(t_1, \ldots, t_a) & \text{iff} \quad (t_1^G, \ldots, t_a^G) \in P^G \\
G & \models & \sim\alpha & \text{iff} \quad G \not\models \alpha \\
G & \models & \alpha \wedge \beta & \text{iff} \quad G \models \alpha \quad \textbf{and} \quad G \models \beta \\
G & \models & \alpha \vee \beta & \text{iff} \quad G \models \alpha \quad \textbf{or} \quad G \models \beta \\
G & \models & \forall x(\alpha) & \text{iff} \quad \textbf{for all } a \in |G| \quad G[a/x] \models \alpha \\
G & \models & \exists x(\alpha) & \text{iff} \quad \textbf{exists } a \in |G| \quad G[a/x] \models \alpha
\end{array}
$$

	introduction	elimination
$\wedge$	$\dfrac{\alpha \quad \beta}{\alpha \wedge \beta}$	$\dfrac{\alpha \wedge \beta}{\alpha}$ $\quad$ $\dfrac{\alpha \wedge \beta}{\beta}$
$\vee$	$\dfrac{\alpha}{\alpha \vee \beta}$ $\quad$ $\dfrac{\beta}{\alpha \vee \beta}$	$\dfrac{\alpha \vee \beta \quad \alpha \vdash \psi \quad \beta \vdash \psi}{\psi}$
$\rightarrow$	$\dfrac{\alpha \vdash \beta}{\alpha \rightarrow \beta}$	$\dfrac{\alpha \rightarrow \beta \quad \alpha}{\beta}$ $\quad$ $\dfrac{\alpha \rightarrow \beta \quad \sim\beta}{\sim\alpha}$
F	$\dfrac{\alpha \quad \sim\alpha}{\mathbf{F}}$	$\dfrac{\alpha \vdash \mathbf{F}}{\sim\alpha}$ $\quad$ $\dfrac{\sim\alpha \vdash \mathbf{F}}{\alpha}$
$\sim\sim$	$\dfrac{\alpha}{\sim\sim\alpha}$	$\dfrac{\sim\sim\alpha}{\alpha}$
$=$	$\overline{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$	$\dfrac{\Gamma \vdash \varphi[x_0/x]}{\Gamma \vdash \forall x\ \varphi}$	$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	$\dfrac{\Gamma \vdash \exists x\ \varphi \quad \Gamma, \varphi[x_0/x] \vdash \psi}{\Gamma \vdash \psi}$

PredCalc Natural Deduction Rules

Proviso for $\forall$-i and $\exists$-e: x_0 is a "new" variable, i.e., it does not appear in φ, ψ, or Γ, i.e., in any visible assumption.

	introduction	elimination
$=$	$\dfrac{}{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$	$\dfrac{\Gamma \vdash \varphi[x_0/x]}{\Gamma \vdash \forall x\ \varphi}$	$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	$\dfrac{\Gamma \vdash \exists x\ \varphi \quad \Gamma, \varphi[x_0/x] \vdash \psi}{\Gamma \vdash \psi}$

1	$\sim\alpha \vee \sim\beta$	
2	$\alpha \wedge \beta$	
3	$\sim\alpha$	
4	α	$\wedge$-e, 2
5	**F**	**F**-i, 3, 4
6	$\sim\beta$	
7	β	$\wedge$-e, 2
8	**F**	**F**-i, 6, 7
9	**F**	$\vee$-e, 1, 3–5, 6–8
10	$\sim(\alpha \wedge \beta)$	**F**-e, 2–9

Theorems, Assumptions, and Proofs

From **assumption** $(\sim\alpha \vee \sim\beta)$, we **proved** $\sim(\alpha \wedge \beta)$.

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There is a **proof** of φ, that **may use assumptions** from the set Γ.

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$\vdash \varphi$: φ is a **theorem** of PredCalc.

There is a **proof** of φ, with no assumptions.

$$\frac{\varphi \;\vdash\; \psi}{\vdash\; \varphi \to \psi} \quad \to\text{-i}$$

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$\vdash \quad \sim\alpha \vee \sim\beta \ \rightarrow \sim(\alpha \wedge \beta)$

Natural Deduction: the Obvious Quantifier Rules

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$\forall$-e $\quad \dfrac{\forall x(\varphi)}{\varphi[t/x]}$ **If $\forall x(\varphi)$ then for any term t, $\varphi[t/x]$**

$\exists$-i $\quad \dfrac{\varphi[t/x]}{\exists x(\varphi)}$ **If I proved $\varphi[t/x]$ then I know $\exists x(\varphi)$**

$\forall$-i

$$\forall\text{-i} \quad \frac{\varphi[x_0/x]}{\forall x(\varphi)}$$

∀-i

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Proviso:
x_0 **does not occur** in φ,
nor in any current assumption

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Informally, we say, “Let x_0 be **arbitrary**.”

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Then we prove $\varphi[x_0/x]$.

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Informally, we say, "Let x_0 be **arbitrary**."

Then we prove $\varphi[x_0/x]$.

We conclude, "Since x_0 was **arbitrary**, it follows that $\forall x(\varphi)$."

Why the **proviso** is **necessary**

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$$\varphi \quad \equiv \quad (x = x_0 \;\vee\; x = s)$$

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1	$x_0 = x_0$	=-i

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$\varphi[x_0/x]$	2	$x_0 = x_0 \lor x_0 = s$	$\lor$-i, 1

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	1	$x_0 = x_0$	$=$-i
$\varphi[x_0/x]$	2	$x_0 = x_0 \lor x_0 = s$	$\lor$-i, 1
$\forall x(\varphi)$	3	$\forall x(x = x_0 \lor x = s)$	$\forall$-i, 2

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$\varphi[x_0/x]$	2	$x_0 = x_0 \ \vee \ x_0 = s$	$\vee$-i, 1
$\forall x(\varphi)$	3	$\forall x(x = x_0 \ \vee \ x = s)$	$\forall$-i, 2 **X X X**

Line 3 **does not follow** because x_0 occurs in φ.

∃-e

∃-e $$\frac{\exists x(\varphi),\ \varphi[x_0/x] \vdash \psi}{\psi}$$

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Informally, we say, “Since we know that something satisfies φ,”

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Informally, we say, "Since we know that something satisfies φ," **we may assume** $\varphi[x_0/x]$ for a **new** variable x_0.

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If we **prove** ψ **using this assumption**

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Informally, we say, "Since we know that something satisfies φ,"

we may assume $\varphi[x_0/x]$ for a **new** variable x_0.

If we **prove** ψ **using this assumption**

and x_0 **does not occur** in ψ

then we **have proved** ψ, with **no assumptions required**.

1	$x = y$	
2	$x = z$	
3	$y = z$	$=$-e, 1, 2
4	$x = z \rightarrow y = z$	$\rightarrow$-i, 2–3
5	$\forall z(x = z \rightarrow y = z)$	$\forall$-i, 4
6	$x = x \rightarrow y = x$	$\forall$-e, 5
7	$x = x$	$=$-i
8	$y = x$	$\rightarrow$-e, 6, 7
9	$x = y \rightarrow y = x$	$\rightarrow$-i, 1–8
10	$\forall xy(x = y \rightarrow y = x)$	$\forall$-i, 9

Is this "proof" of $\vdash \exists x \forall y E(x,y) \rightarrow \forall y \exists x E(x,y)$ correct?

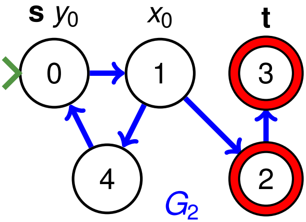
1	$\exists x \forall y E(x,y)$	
2	$\forall y E(x_0,y)$	
3	$E(x_0,y_0)$	$\forall$-e, 2
4	$\exists x E(x,y_0)$	$\exists$-i, 3
5	$\exists x E(x,y_0)$	$\exists$-e, 1, 2–4
6	$\forall y \exists x E(x,y)$	$\forall$-i, 5
7	$\exists x \forall y E(x,y) \rightarrow \forall y \exists x E(x,y)$	$\rightarrow$-i, 1–6

Is this "proof" of $\vdash \exists x \forall y E(x, y) \rightarrow \forall y \exists x E(x, y)$ correct? **Yes.**

1	$\exists x \forall y E(x, y)$	
2	$\forall y E(x_0, y)$	
3	$E(x_0, y_0)$	$\forall$-e, 2
4	$\exists x E(x, y_0)$	$\exists$-i, 3
5	$\exists x E(x, y_0)$	$\exists$-e, 1, 2–4
6	$\forall y \exists x E(x, y)$	$\forall$-i, 5
7	$\exists x \forall y E(x, y) \rightarrow \forall y \exists x E(x, y)$	$\rightarrow$-i, 1–6

Is this “proof” of $\vdash \forall y \exists x E(x, y) \rightarrow \exists x \forall y E(x, y)$ correct?

1	$\forall y \exists x E(x, y)$	
2	$\exists x E(x, y_0)$	$\forall$-e, 1
3	$E(x_0, y_0)$	☐, 2
4	$\forall y E(x_0, y)$	☐, 3
5	$\exists x \forall y E(x, y)$	☐, 4
6	$\forall y \exists x E(x, y) \rightarrow \exists x \forall y E(x, y)$	☐, 1–5



Is this “proof” of $\vdash \forall y \exists x E(x,y) \rightarrow \exists x \forall y E(x,y)$ correct?

No. Line 3 does not follow from lines 1 and 2.

$G_2 \models$ lines 1 and 2

$G_2 \models \sim E(x_0, y_0)$

1	$\forall y \exists x E(x,y)$	
2	$\exists x E(x, y_0)$	$\forall$-e, 1
3	$E(x_0, y_0)$	[ ], 2
4	$\forall y E(x_0, y)$	[ ], 3
5	$\exists x \forall y E(x,y)$	[ ], 4
6	$\forall y \exists x E(x,y) \rightarrow \exists x \forall y E(x,y)$	[ ], 1–5

1. Convert to NNF

a. A4 $\sim\forall x\ \exists y\ (A(x) \rightarrow E(x,y))$

b. A6 $\sim\exists x\ \exists y\ (E(x,y) \wedge \sim(A(x) \wedge R(y)))$

c. A5 $\sim\forall x\ \exists y\ (R(x) \rightarrow E(y,x))$

d. A3 $\sim\exists x\ \exists y\ \exists z\ (E(x,y)\ \wedge\ E(y,z)\ \wedge\ y \neq z)$

Possible Answers:

A1. $\forall x\ \exists y\ (R(x) \wedge\ \sim E(y,x))$

A2. $\forall xyz\ (\sim E(x,y) \vee\ \sim E(y,z) \rightarrow y = z)$

A3. $\forall xyz\ (\sim E(x,y) \vee\ \sim E(y,z)\ \vee\ y = z)$

A4. $\exists x\ \forall y\ (A(x) \wedge\ \sim E(x,y))$

A5. $\exists x\ \forall y\ (R(x) \wedge\ \sim E(y,x))$

A6. $\forall x\ \forall y (\sim E(x,y)\ \vee\ (A(x) \wedge R(y)))$

2. Match English statements with PredCalc formulas.

a. B7 s is an A vertex but not an R vertex.

b. B3 Every vertex is either A or R, but not both.

c. B1 Every edge goes from an A vertex to an R vertex.

d. B2 Every R vertex has an incoming edge from an A vertex.

e. B4 Some R vertex has an incoming edge from every A vertex.

Possible Answers:

B1. $\forall x\,\forall y\,(E(x,y) \;\rightarrow\; A(x) \wedge R(y))$

B2. $\forall y\,\exists x\,(R(y) \;\rightarrow\; A(x) \wedge E(x,y))$

B3. $\forall x\,(A(x) \oplus R(x))$

B4. $\exists y\,\forall x\,(R(y) \;\wedge\; (A(x) \rightarrow E(x,y))$

B5. $\forall x\,((A(x) \vee R(x)) \;\wedge\; (A(x) \leftrightarrow R(x)))$

B6. $A(s) \;\rightarrow \sim R(s)$

B7. $A(s) \wedge \;\sim R(s)$

B8. $\exists y\,\forall x\,(R(y) \;\rightarrow\; A(x) \;\rightarrow\; E(x,y))$

3.1 Is this "proof" of $\vdash R(x) \wedge A(y) \rightarrow A(y) \wedge R(x)$ correct?

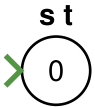
1	$R(x) \wedge A(y)$	
2	$R(x)$	$\wedge$-e, 1
3	$A(y)$	$\wedge$-e, 1
4	$A(y) \wedge R(x)$	$\wedge$-i, 2, 3
5	$R(x) \wedge A(y) \rightarrow A(y) \wedge R(x)$	$\rightarrow$-i, 1–4

This proof is correct.

3.2 Is this "proof" of $(\forall x \,.\, R(x))(A(x)) \vdash (\exists x \,.\, R(x))(A(x))$ correct?

1	$\forall x\,(R(x) \to A(x))$	
2	$R(x) \to A(x)$	$\forall$-e, 1
3	$R(x)$	
4	$A(x)$	$\to$-e, 2, 3
5	$R(x) \wedge A(x)$	$\wedge$-i, 3, 4
6	$A(x)$	
7	$R(x) \wedge A(x)$	X, 3, 6
8	$R(x) \wedge A(x)$	X, 2, 3–5, 6–7
9	$\exists x\,(R(x) \wedge A(x))$	$\exists$-i, 8

No, $G_7 \models$ lines 1 and 6, but not line 7 or 8.



G_7

4. Match PredCalc, English and graphs.

a.	E2	G_4	$\exists x \, \forall y \, (E(x,y) \vee E(y,x) \rightarrow x = y)$
b.	E3	G_2	$\exists y \, \forall x \, (A(y) \wedge \sim E(x,y))$
c.	E4	G_3	$\forall xy \, (E(x,y) \rightarrow E(y,x))$
d.	E1	G_1, G_2	$\exists! x \, \exists! y \, E(x,y)$

E1. Exactly one vertex has outdegree exactly 1.

E2. Some vertex is isolated, i.e., has no edge in or out to a different vertex

E3. Some A vertex has no incoming edge.

E4. The graph is undirected.