COMPSCI 614: Randomized Algorithms with Applications to Data Science

Prof. Cameron Musco

University of Massachusetts Amherst. Spring 2024.

Lecture 6

Logistics

- Problem Set 1 is due tomorrow at midnight.
- I am holding office hours directly after class today.
- No class or office hours on Thursday.
- Problem Set 2 will be posted later this week.
- Quiz due Monday

Summary

Last Time:

- Stronger concentration bounds for sums of independent random variables. I.e., exponential concentration bounds.
- Chernoff and Bernstein bound.
- Application to balls-into-bins and linear probing analysis.

$$\frac{\log n}{\log \log n}$$

Summary

Last Time:

- Stronger concentration bounds for sums of independent random variables. I.e., exponential concentration bounds.
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- Application to balls-into-bins and linear probing analysis.

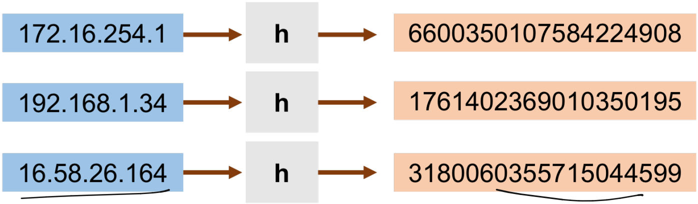
Today:

- Random hash functions and fingerprinting.
- Applications to pattern matching and communication complexity.

Random Hashing and Fingerprinting

Random Hash Functions

A random hash function maps inputs to random outputs.



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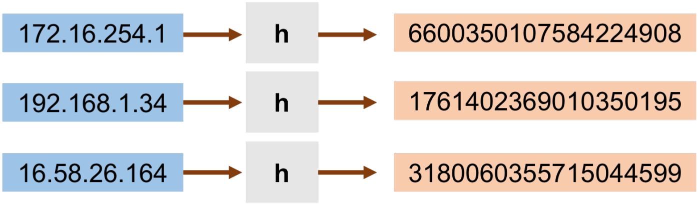

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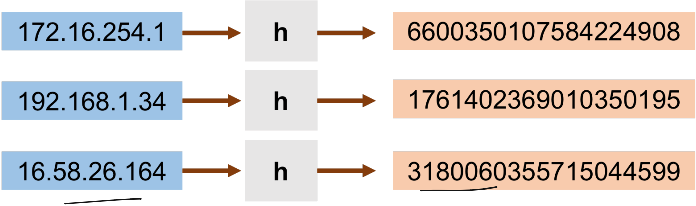

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```
import random
a = random.randint(1,100)
b = random.randint(1,100)
def myHash(x):
  return (a*x+b) % 100
```

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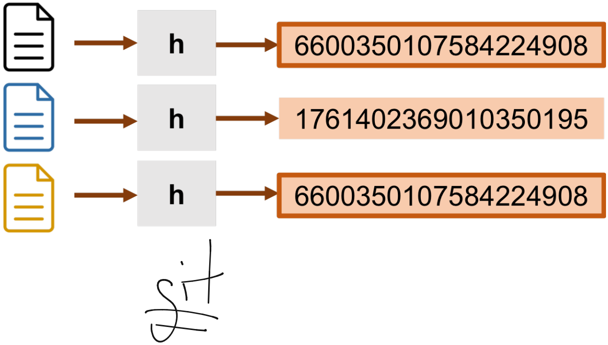
Fingerprinting

Random hash functions are often used to reduce large files down to hash 'fingerprints', which can be used to check equality of files (deduplication), detect updates/corruptions, etc.



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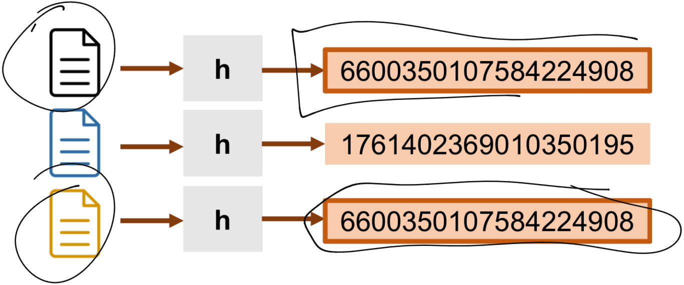

- Key requirement is that two distinct files are unlikely to have the same hash – low collision probability.

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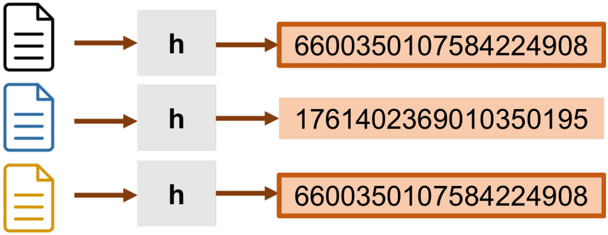

- Key requirement is that two distinct files are unlikely to have the same hash – low collision probability.
- In practice h is often a deterministic 'cryptographic' hash function like SHA or MD5 – hard to analyze formally.

Rabin Fingerprint

Rabin Fingerprint: Interpret a bit string $x_1, x_2, \ldots, x_n$ as the binary representation of the integer $x = \sum_{i=1}^{n} x_i \cdot 2^{i-1}$. Let

$$\mathrm{h}(x) = x \mod p,$$

where p is a randomly chosen prime in $[1, tn \log tn]$.

Prime Number Theorem: There are $\approx \frac{tn \log tn}{\log(tn \log tn)} = \Theta(tn)$ primes in $[1, tn \log tn]$. So p is chosen randomly from $\Theta(tn)$ possible values.

Claim: For $x, y \in [0, 2^n]$ with $x \neq y$, $\Pr[\mathrm{h}(x) = \mathrm{h}(y))] = O(1/t)$.

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- If $\mathrm{h}(x) = \mathrm{h}(y)$, then it must be that $x - y \mod p = 0$. I.e., p divides $x - y$. So we must bound the probability of this occuring.

$x \equiv y \mod p$

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- If $\mathbf{h}(x) = \mathbf{h}(y)$, then it must be that $x - y \mod p = 0$. I.e., p divides $x - y$. So we must bound the probability of this occuring.
- **Note:** This is not a cryptographic hash function – it is relatively easy to find x, y with $\mathbf{h}(x) = \mathbf{h}(y)$ given p, or blackbox access to h. However, this is fine in many applications.

Rabin Fingerprint Analysis

$x - y \in [-2^n, 2^n]$ $\quad 2 \cdot 3 \cdot 5 \cdot$

Think-Pair-Share 1: How many unique prime factors can an integer in $[-2^n, 2^n]$ have? $\leq n$

$120 = 2^3 \cdot 3 \cdot 5$

$m = 2^{i_2} \cdot 3^{i_3} \cdot 5^{i_5} \cdots$

$m \geq 2^{\#\text{unique factors}} \to \#\text{unique factors} \leq \log_2 m = n$

Think-Pair-Share 2: What is the probability that a random prime p chosen from $[1, tn \log tn]$ divides $x - y \in [-2^n, 2^n]$? I.e., that $h(x) = h(y)$? Recall: There are $\Theta(tn)$ primes in the range $[1, tn \log tn]$.

$m = 2 \cdot 3 \cdot 7 \cdot 11^2$ — n unique primes

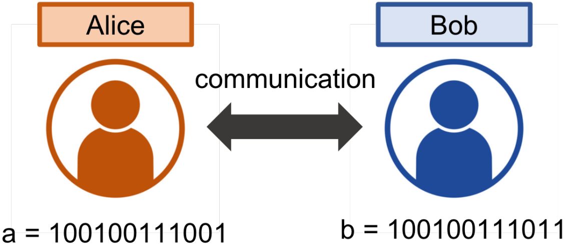
$x - y \bmod p = m \bmod p = 0$ iff $p =$ one of the unique factors of m

$\Pr[h(x) = h(y)] = \frac{\text{factors in PF of } x-y}{\#\text{ possible vals for } p} \leq \frac{n}{\Theta(tn)} = \Theta\left(\frac{1}{t}\right)$

Fingerprinting Application 1: Communication Complexity

Fingerprinting for Equality Testing

Equality Testing Communication Problem: Alice has some bit string $a \in \{0,1\}^n$. Bob has some string $b \in \{0,1\}^n$. How many bits do they need to communicate to determine if $a = b$ with probability at least $2/3$?



Fingerprinting for Equality Testing

Equality Testing Protocol:

- Alice picks a random prime $p \in [1, tn \log tn]$ for some large constant t.
- Alice sends p, along with the Rabin fingerprint $\mathsf{h}(a) := a \mod p$ to Bob.
- Bob uses p to compute $\mathsf{h}(b) := b \mod p$.
- If $\mathsf{h}(a) = \mathsf{h}(b)$, Bob sends 'YES' to Alice. Else, he sends 'No'.

Fingerprinting for Equality Testing

deterministically by picking
until values for p → O(n log n)

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$a - b \mod p = 0$

Correctness: If $a = b$ both Alice and Bob always output 'YES'. If $a \neq b$ they output 'NO' with probability $1 - O(1/t) \geq 2/3$ if t is set large enough.

Runtime: "word-RAM" model = O(1) runtime.
"actual asymptotic runtime" = O(# bits integers)
= O(log(n/d))

Fingerprinting for Equality Testing

Equality Testing Protocol:

$\log(n) + \log\log n \leq 2\log n$

- Alice picks a random prime $p \in [1, tn \log tn]$ for some large constant t.

  $\log p = \log tn \log tn = O(\log(n \log n)) = O(\log n)$
- Alice sends p, along with the Rabin fingerprint $\mathbf{h}(a) := a \mod p$ to Bob.

  $O(\log n)$
- Bob uses p to compute $\mathbf{h}(b) := b \mod p$.
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  $O(1)$

Correctness: If $a = b$ both Alice and Bob always output 'YES'. If $a \neq b$ they output 'NO' with probability $1 - O(1/t) \geq 2/3$ if t is set large enough.

success probability $1-\delta$ for tiny δ

Complexity: $O(\log(n))$

repeating: $O(\log n \cdot \log(1/\delta))$ | pick bigger $t = O(1/\delta)$ $O(\log(nt)) = O(\log(n/\delta))$

Fingerprinting for Equality Testing

Equality Testing Protocol:

- Alice picks a random prime $p \in [1, tn \log tn]$ for some large constant t.
- Alice sends p, along with the Rabin fingerprint $\mathbf{h}(a) := a \mod p$ to Bob. [$O(\log p) = O(\log n)$ bits]
- Bob uses p to compute $\mathbf{h}(b) := b \mod p$.
- If $\mathbf{h}(a) = \mathbf{h}(b)$, Bob sends 'YES' to Alice. Else, he sends 'No'. [1 bit]

Correctness: If $a = b$ both Alice and Bob always output 'YES'. If $a \neq b$ they output 'NO' with probability $1 - O(1/t) \geq 2/3$ if t is set large enough.

Complexity: Uses just $O(\log p) = O(\log n)$ bits of communication in total.

Deterministic Equality Testing

How many bits must Alice and Bob send if they want to check equality of $a, b \in \{0, 1\}^n$ without using randomness?

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- An exponential separation between randomized and deterministic protocols!
- Unlike for running times, for communication complexity problems there are often large provable separations between randomized and deterministic protocols.

Deterministic Equality Testing Lower Bound

Claim: Any deterministic protocol for equality testing requires sending $\Omega(n)$ bits.

- Assume without loss of generality that Alice and Bob alternate sending 1 bit at a time – at most doubles the number of bits.

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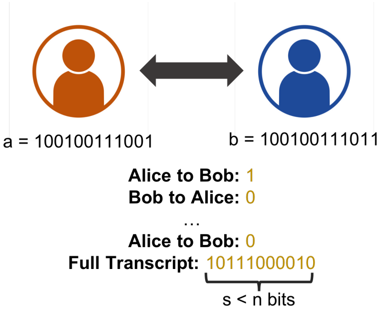
Claim: Any deterministic protocol for equality testing requires sending $\Omega(n)$ bits.

- Assume without loss of generality that Alice and Bob alternate sending 1 bit at a time – at most doubles the number of bits.
- If Alice and Bob send $s < n$ bits, in total, there are 2^s possible conversations they may have.



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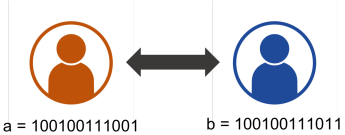

Alice to Bob: 1
Bob to Alice: 0
…
Alice to Bob: 0
Full Transcript: 10111000010

s < n bits

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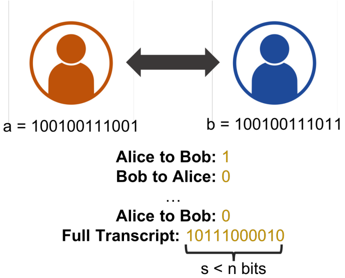

- Since there are $2^n > 2^s$ possible inputs, there must be two different inputs $v_1 \neq v_2$, such that given $a = b = v_1$ or $a = b = v_2$, the protocol outputs 'YES' and has identical transcripts.

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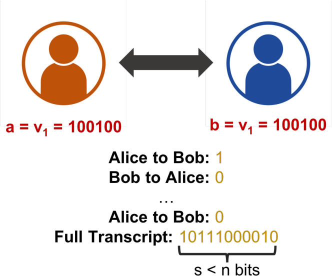

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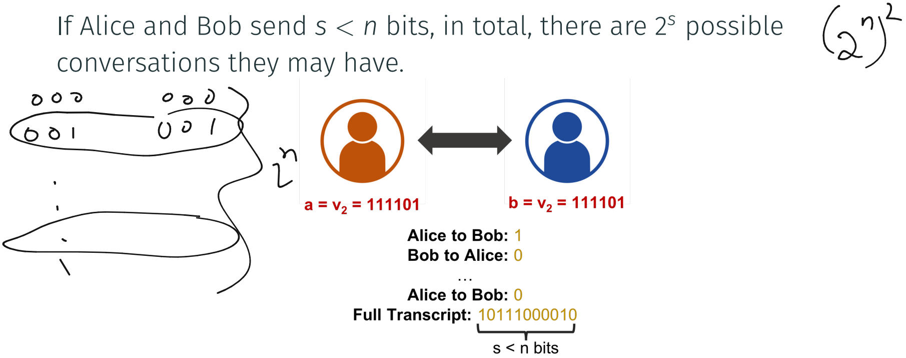

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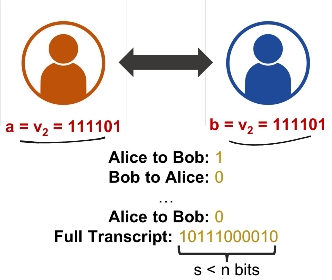

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- But then the players will send the same messages and output 'YES' also when Alice is given $a = v_1$ and Bob is given $b = v_2$. This violates correctness!

Deterministic Equality Testing Lower Bound

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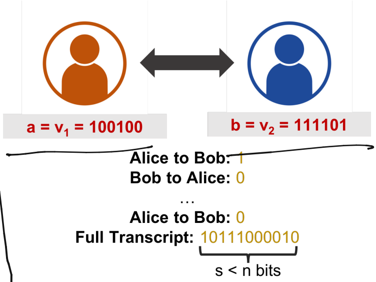
EQ equality
GE greater than
DISJ set disjointness



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Application 2: Pattern Matching

Pattern Matching

Given some document $x = x_1 x_2 \ldots x_n$ and a pattern $y = y_1 y_2 \ldots y_m$, find some j such that

$m << n$

$$x_j x_{j+1}, \ldots, x_{j+m-1} = y_1 y_2 \ldots y_m.$$

x = The quick brown **fox** jumped across the pond…

y = fox

Can assume without loss of generality that the strings are binary strings.

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What is the ‘naive’ running time required to solve this problem?

$O(nm)$ - double loop

$O(n+m)$ is optimal can be done deterministically even.

Rolling Hash

We will use the fact that the Rabin fingerprint is a rolling hash.

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- Letting $X_j = \sum_{i=0}^{m-1} x_{j+i} \cdot 2^{m-1-i}$ be the integer value represented by the binary string $x_j x_{j+1}, \ldots, x_{j+m-1}$, we have

$$X_{j+1} = 2 \cdot X_j - 2^m x_j + x_{j+m}.$$

m = 3

0 0 1 0 1 0 1 1

$X_0 = 1 \quad X_1 = 2 \quad X_2 = 5 \ldots$

$X_0 = 1 \cdot 2 + 0 + 0 = 2$

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$$X_{j+1} = 2 \cdot X_j - 2^m x_j + x_{j+m}.$$

- Thus, since for any X, $\mathbf{h}(X) = X \mod p$,

$$\mathbf{h}(X_{j+1}) = 2 \cdot \mathbf{h}(X_j) - 2^m x_j + x_{j+m} \mod p.$$

- Given $\mathbf{h}(X_j)$, this hash value can be computed using just $O(1)$ arithmetic operations.

Rabin-Karp Algorithm

The Rabin-Karp pattern matching algorithm is then:

- Pick a random prime $p \in [1, tm \log mt]$, for $t = cn$.
- Let $Y = \mathrm{h}(y)$ be the Rabin fingerprint of the pattern.
- Let $H = \mathrm{h}(X_1)$ be the Rabin fingerprint of the first block of text.
- For $j = 1, \ldots, x_{n-m+1}$
  - If $Y == H$, return j.
  - Else, $H = 2 \cdot H - 2^m x_j + x_{j+m} \mod p$.

O(1)

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Runtime: Takes $O(m+n)$ time in total. $O(m)$ for the initial hash computations, and $O(1)$ for each iteration of the for loop.

Correctness: The probability of a false positive at any step is upper bounded by $\frac{1}{t} = \frac{1}{cn}$. Thus, via a union bound, the probably of a false positive overall is at most $\frac{n}{cn} = \frac{1}{c}$.