COMPSCI 614: Randomized Algorithms with Applications to Data Science

Prof. Cameron Musco

University of Massachusetts Amherst. Spring 2024.

Lecture 3

Logistics

- Reminder that there is a weekly quiz, released after class today and due next Monday 8pm.
- Problem Set 1 will be released shortly – hopefully by the end of the week. Sorry for the delay.
- See Piazza for a post to organize homework groups.

Summary

Last Time:

- Review of conditional probability, independence, linearity of expectation and variance.
- Polynomial identity testing and proof of the Schwartz-Zippel Lemma. $p(z_1, \ldots, z_n)$ $z_i \in S$
- Application of linearity of expectation to randomized Quicksort analysis. $O(n \log n)$

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- Review of conditional probability, independence, linearity of expectation and variance.
- Polynomial identity testing and proof of the Schwartz-Zippel Lemma.
- Application of linearity of expectation to randomized Quicksort analysis.

Today:

- Concentation bounds – Markov’s and Chebyshev’s inequalities.
- The union bound.
- Applications to coupon collecting and statistical estimation.

Concentration Inequalities

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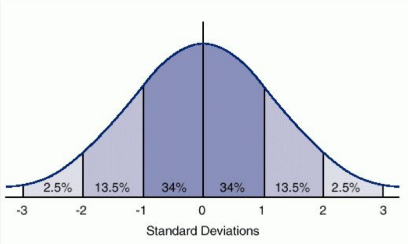
Concentration inequalities are bounds showing that a random variable lies close to it's expectation with good probability. Key tools in the analysis of randomized algorithms.



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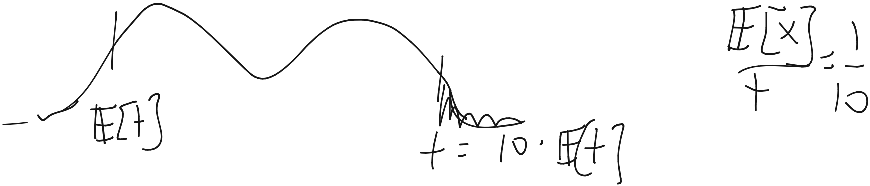

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$$\geq \sum_{u \geq t} \Pr(X = u) \cdot t$$

$$= t \cdot \sum_{u \geq t} P(X = u)$$

$$= t \cdot \Pr(X \geq t)$$

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Plugging in $t = \mathbb{E}[X] \cdot s$, $\Pr[X \geq s \cdot \mathbb{E}[X]] \leq 1/s$. The larger the deviation s, the smaller the probability.

ZPP ⊆ BPP

Think-Pair-Share: You have a Las Vegas algorithm that solves some decision problem in expected running time T. Show how to turn this into a Monte-Carlo algorithm with worst case running time $3T$ and success probability 2/3.

$$\Pr(\text{runtime} \geq 3 \cdot \mathbb{E}[\text{runtime}]) \leq \frac{1}{3}$$

$$\Pr(\text{runtime} \geq 3T) \leq \frac{1}{3}$$

After 3T steps: terminate + guess

↳ succeeds at least 2/3 of the time.

Chebyshev's inequality

With a very simple twist, Markov's Inequality can be made much more powerful in many settings.

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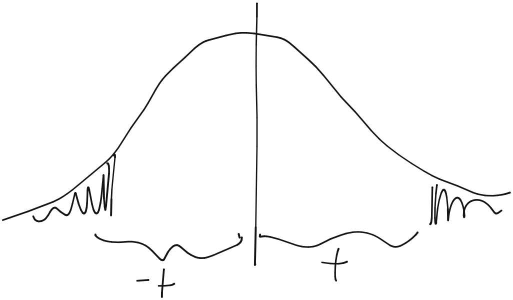
$$\Pr(|X| \geq t) = \Pr(X^2 \geq t^2).$$



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Plugging in the random variable $X - \mathbb{E}[X]$, gives the standard form of **Chebyshev's inequality:**

$$\Pr(|X - \mathbb{E}[X]| \geq t) \leq \frac{\mathbb{E}[(X - \mathbb{E}[X])^2]}{t^2} = \frac{\text{Var}(X)}{t^2}.$$

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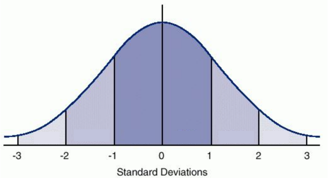
What is the probability that X falls s standard deviations from it's mean?



s.d.

$$\Pr(|X - \mathbb{E}[X]| \geq s \cdot \sqrt{\text{Var}[X]}) \leq \frac{\text{Var}[X]}{s^2 \cdot \text{Var}[X]} = \frac{1}{s^2}.$$

Application 2: Statistical Estimation + Law of Large Numbers

Concentration of Sample Mean

Theorem: Let $X_1, \ldots, X_n$ be pairwise independent random variables with $\mathbb{E}[X_i] = \mu$ and $\text{Var}[X_i] = \sigma^2$. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_n$ be their sample average.

For any $\epsilon > 0$, $\Pr[|\bar{X} - \mu| \geq \epsilon\sigma] \leq \frac{1}{n\epsilon^2}$.

$n \to \infty \qquad \Pr[\ ] \to 0$

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- By linearity of expectation, $\mathbb{E}[\bar{X}] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i] = \mu$.
- By linearity of variance, $\mathbb{E}[\bar{X}] = \frac{1}{n^2}\sum_{i=1}^{n} \text{Var}[X_i] = \frac{\sigma^2}{n}$.

Var

$$\text{Var}(\bar{X}) = \text{Var}\left(\frac{1}{n}\sum X_i\right)$$

$$= \frac{1}{n^2}\sum \text{Var}(X_i)$$

$$= \frac{1}{n^2} \cdot n \cdot \sigma^2 = \frac{\sigma^2}{n}$$

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- Plugging into Chebyshev's inequality:

σ^2/n

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This is the weak law of large numbers.

Concentration of Sample Mean

Application to statistical estimation: There is a large population of individuals. A p fraction of them have a certain property (e.g., 55% of people support decreased taxation, 10% of people are greater than 6' tall, etc.). Want to estimate p from a small sample of individuals.

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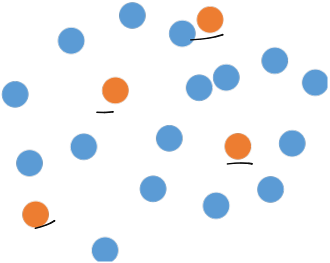

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- Sample n individuals uniformly at random, with replacement.
- Let $X_i = 1$ if the i^{th} individual has the property, and 0 otherwise. $X_1, \ldots, X_n$ are i.i.d. draws from $Bern(p)$ – each is 1 with probability p and 0 with probability $1 - p$.

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- $\mathbb{E}[X_i] = p$ and $\text{Var}[X_i] = p(1-p)$.
- Thus, letting $\bar{p} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\mathbb{E}[\bar{p}] = p$ and $\text{Var}[\bar{p}] = \frac{p(1-p)}{n} \leq \frac{p}{n}$.

fraction of samples w/ property

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- By Chebyshev's inequality $\Pr[|p - \bar{p}| \geq \epsilon] \leq \frac{p}{\epsilon^2 n}$.

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- By Chebyshev's inequality $\Pr[|p - \bar{p}| \geq \epsilon] \leq \frac{p}{\epsilon^2 n}$.

Upshot: If we take $n = \frac{p}{\epsilon^2 \delta}$ samples, then with probability at least $1 - \delta$, $\bar{p}$ will be a $\pm\epsilon$ estimate to the true proportion p. A prototypical sublinear time algorithm.

Application to Success Boosting

$O(T \log(1/\delta))$ BPP $2/3$

Think-Pair-Share: You have a Monte-Carlo algorithm with worst case running time T and success probability $2/3$. Show how to obtain, for any $\delta \in (0,1)$, a Monte-Carlo algorithm with worse case running time $O(T/\delta)$ and success probability $1-\delta$.

run n times $n = O(1/\delta)$

return majority

$X_1 \dots X_n$ $= 1$ if correct, 0 if incorrect

$\frac{1}{n}\sum X_i > \frac{1}{2}$

$\bar{p} > \frac{1}{2}$

$\mathbb{E}[X_i] = p = 2/3$

$\bar{p} \geq \frac{1}{2}$

$|p - \bar{p}| < \frac{1}{6}$

$\frac{P}{n \cdot \varepsilon^2} \leq \delta$

$\frac{2/3}{n \cdot 1/6^2} \leq \delta$

$n \geq \frac{1}{\delta} \cdot C$

Application 3: Coupon Collecting

Coupon Collector Problem

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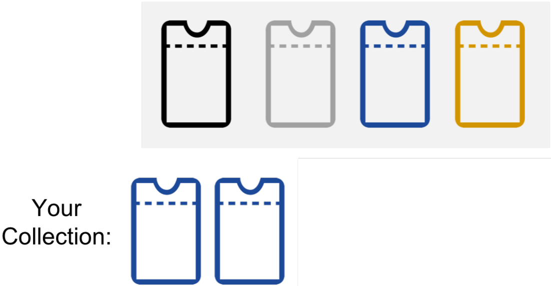

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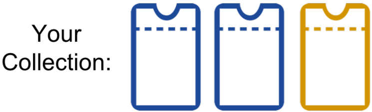

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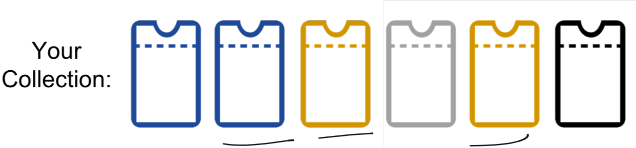

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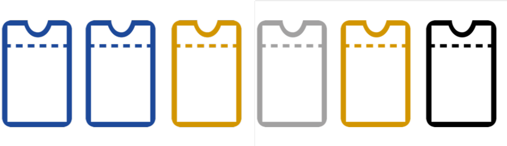

Think-Pair-Share: Say you have collected i coupons so far. Let T_{i+1} denote the number of draws needed to collect the $(i+1)^{st}$ coupon. What is $\mathbb{E}[T_i]$? $\frac{n}{n-i}$

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- $\mathbf{T}_i$ is a geometric random variable with success probability $p_i = \frac{n-i}{n}$. I.e., $Pr[\mathbf{T}_i = j] = p_i(1-p_i)^{j-1}$.
- **Exercise:** verify that $\mathbb{E}[\mathbf{T}_i] = 1/p_i = \frac{n}{n-i}$.

$$\mathbb{E}\, T_i = p_i \cdot 1 + (1-p_i)\cdot(\mathbb{E}[T_i]+1)$$

$$\mathbb{E}[T_i] = p_i + 1 - p_i + (1-p_i)\cdot \mathbb{E}[T_i]$$

$$p_i\, \mathbb{E}[T_i] = 1$$

$$\mathbb{E}[T_i] = 1/p_i$$

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$$\mathbb{E}[T] = \sum_{i=0}^{n-1} \mathbb{E}[T_i] = \frac{n}{n} + \frac{n}{n-1} + \ldots \frac{n}{2} + \ldots \frac{n}{1}$$

$$= n \cdot \left(\frac{1}{n} + \frac{1}{n-1} + \ldots + 1 \right)$$

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$$= n \cdot H_n. \simeq O(n \log n)$$

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- By Markov's inequality, $\Pr[\mathrm{T} \geq cn \cdot H_n] \leq$ 1/c

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$\frac{1}{p_i}$

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$$\text{Var}[\mathsf{T}] = \sum_{i=0}^{n} \frac{1 - p_i}{p_i^2} = \sum_{i=0}^{n} \frac{1}{p_i^2} - \sum_{i=0}^{n} \frac{1}{p_i}$$

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$$\leq n^2 \cdot \frac{\pi^2}{6} - n \cdot H_n$$

$$\sum_{x=0}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6}$$

$$\sum_{i=0}^{n-1} \frac{1}{p_i^2} = \sum_{i=0}^{n-1} \frac{n^2}{(n-i)^2} = n^2 \sum_{i=1}^{n} \frac{1}{i^2} \leq n^2 \frac{\pi^2}{6}$$

Coupon Collector Analysis

Can get a tighter tail bound using Chebyshev's inequality in place of Markov's.

- We wrote $\mathsf{T} = \sum_{i=0}^{n-1} \mathsf{T}_i$, which let us compute $\mathbb{E}[\mathsf{T}] = n \cdot H_n$.
- Also have $\mathsf{Var}[\mathsf{T}] = \sum_{i=0}^{n-1} \mathsf{Var}[\mathsf{T}_i]$. Why?
- **Exercise:** show that $\mathsf{Var}[\mathsf{T}_i] = \frac{1-p_i}{p_i^2}$, and recall that $p_i = \frac{n-i}{n}$.
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$\mathbb{E}[T]$

nlogn + cn

nlogn

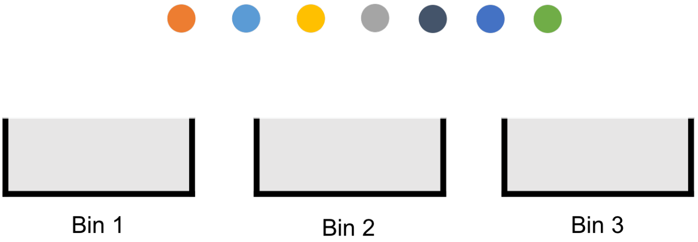
≤ cnlogn

- Via Chebyshev's inequality, $\Pr[|\mathrm{T} - n \cdot H_n| \geq cn] \leq \frac{n^2 \, \pi^2/6}{c^2 n^2} = \frac{\pi^2/6}{c^2}$

Application 4: Randomized Load Balancing and Hashing, and ‘Ball Into Bins’

Balls Into Bins

I throw m balls independently and uniformly at random into n bins. What is the maximum number of balls any bin?



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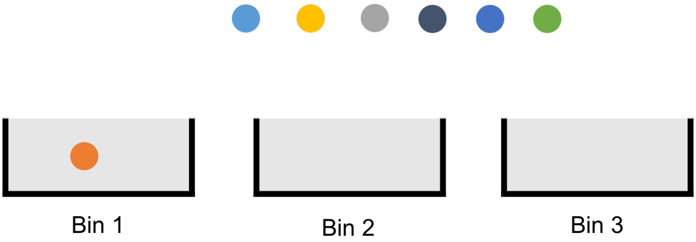

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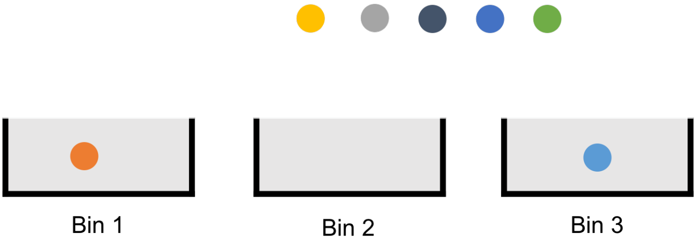

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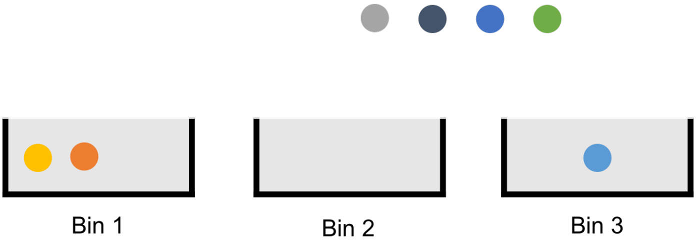

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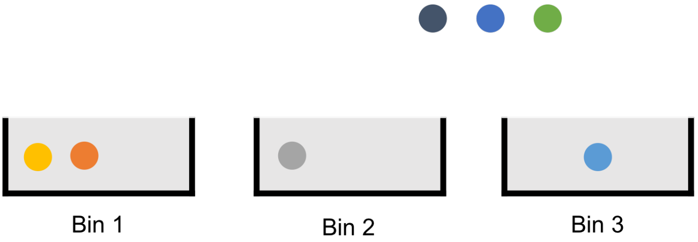

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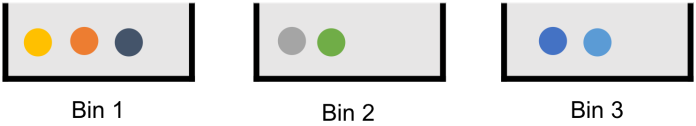
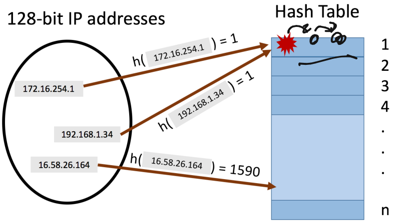


Application: Hash Tables



- **hash function** $h : U \to [n]$ maps elements to indices of an array.
- Repeated elements in the same bucket are stored as a linked list – 'chaining'.
- Worse-case look up time is proportional to the maximum list length – i.e., the maximum number of 'balls' in a 'bin'.

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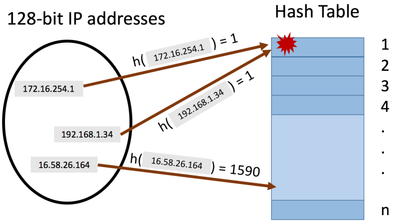

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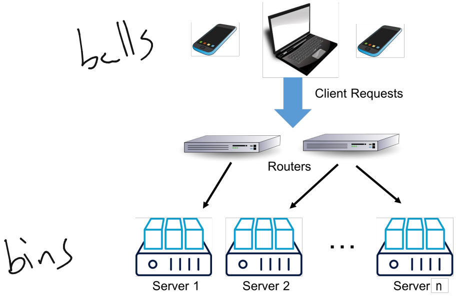
Note: A 'fully random hash function' maps items independently and uniformly at random to buckets. This is a theoretical idealization of practical hash functions.

Application: Randomized Load Balancing



- *m* requests are distributed randomly to *n* servers. Want to bound the maximum number of requests that a single server must handle.
- Assignment is often is done via a random hash function so that repeated requests or related requests can be mapped to the same server, to take advantages of caching and other optimizations.

Balls Into Bins Analysis

Let $\mathbf{b}_i$ be the number of balls landing in bin i. For n balls into m bins what is $\mathbb{E}[\mathbf{b}_i]$? $= \frac{n}{m}$

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where A_i is the event that $\mathbf{b}_i \geq k$.

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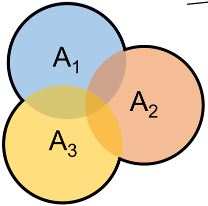
Union Bound: For any random events $A_1, A_2, \ldots, A_n$,

$$\Pr(A_1 \cup A_2 \cup \ldots \cup A_n) \leq \Pr(A_1) + \Pr(A_2) + \ldots + Pr(A_n).$$



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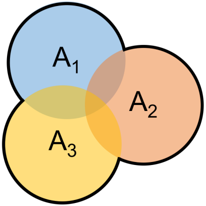

Exercise: Show that the union bound is a special case of Markov's inequality with indicator random variables.

Balls Into Bins Direct Analysis

Let $\mathbf{b}_i$ be the number of balls landing in bin i. If we can prove that for any i, $\Pr[A_i] = \Pr[\mathbf{b}_i \geq k] \leq p$, then by the union bound:

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$\# b_i = 1$

$$\Pr\left[\max_{i=1,\ldots,n} \mathbf{b}_i \geq k\right] = \Pr\left[\bigcup_{i=1}^{n} A_i\right] \leq n \cdot p. \leq \frac{1}{n^2}$$

Claim 1: Assume $m = n$. For $k \geq \frac{c \ln n}{\ln \ln n}$, $\Pr[\mathbf{b}_i \geq k] \leq \frac{1}{n^{c-o(1)}}$.

$k = \frac{3 \ln n}{\ln \ln n}$ $\qquad \Pr[b_i \geq k] \leq \frac{1}{n^3}$

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- We have $\binom{n}{j} \leq \left(\frac{en}{j}\right)^j$, giving $\Pr[\mathbf{b}_i = j] \leq \left(\frac{e}{j}\right)^j \cdot (1 - \frac{1}{n})^{n-j} \leq \left(\frac{e}{j}\right)^j$.

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- Summing over $j \geq k$ we have:

$$\Pr[\mathbf{b}_i \geq k] \leq \sum_{j \geq k} \left(\frac{e}{j}\right)^j \leq \left(\frac{e}{k}\right)^k \cdot \frac{1}{1 - e/k}.$$

$\leq \sum_{j \geq k} \left(\frac{e}{k}\right)^j$

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We just showed: When $n = m$ (i.e., n balls into n bins)

$$\Pr[b_i \geq k] \leq \left(\frac{e}{k}\right)^k \cdot \frac{1}{1 - e/k}$$

For $k = \frac{c \ln n}{\ln \ln n}$ we have:

$$\Pr[b_i \geq k] \leq \left(\frac{\ln \ln n}{\ln n}\right)^{\frac{c \ln n}{\ln \ln n}} \cdot \frac{1}{1 - (e \ln \ln n)/(c \ln n)}$$

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Upshot: By the union bound, For $k = c\frac{\ln n}{\ln \ln n}$ for sufficiently large c,

$$\Pr\left[\max_{i=1,\ldots,n} \mathbf{b}_i \geq k\right] \leq n \cdot \frac{1}{n^{c-o(1)}} = \frac{1}{n^{c-1-o(1)}}.$$

When throwing n balls in to n bins, with very high probability the maximum number of balls in a bin will be $O\left(\frac{\ln n}{\ln \ln n}\right)$.