COMPSCI 614: Randomized Algorithms with Applications to Data Science

Prof. Cameron Musco

University of Massachusetts Amherst. Spring 2024.

Lecture 2

Summary

Last Class:

- Course logistics/overview of planned content.
- Intro to randomized algorithms: Las Vegas vs. Monte Carlo
- Randomized complexity classes including RP, ZPP, BPP, PP.

$> \frac{1}{2}$

one-sided error

always correct
runtime random

correct
w/ good
prob.

RP
ZPP
BPP
PP

$ZPP \subseteq RP \subseteq BPP \subseteq PP$

$NP \subseteq$

Summary

Last Class:

- Course logistics/overview of planned content.
- Intro to randomized algorithms: Las Vegas vs. Monte Carlo
- Randomized complexity classes including RP, ZPP, BPP, PP.

This Class: Basic probability review with algorithmic applications.

- Conditional probability, Baye's theorem, and independence. Application to polynomial identity testing.
- Linearity of expectation and variance. Application to randomized Quicksort analysis.
- Maybe start on concentration inequalities (Markov's and Chebyshev's).

Basic Probability Review

Conditional Probability Review

Consider two random events A and B.

B = die roll is even
A = die role is "2"

- **Conditional Probability:**

$P(A|B) \cdot P(B) = Pr(A \cap B)$

$$\boxed{\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.} \rightarrow \frac{1/6}{1/2} = \frac{1}{3}$$

- **Baye's Theorem:**

$P(B \cap A)$

$$\Pr(B|A) = \frac{\Pr(A|B) \cdot P(B)}{P(A)}.$$

- **Independence:** A and B are independent if:

$$\Pr(A|B) = \Pr(A).$$

Using the definition of conditional probability, independence means:

$$P(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \Pr(A) \implies \boxed{\Pr(A \cap B) = \Pr(A) \cdot \Pr(B).}$$

Independence

Sets of events: For a set of n events, $A_1, \ldots, A_n$, the events are k-wise independent if for any subset S of at most k events,

$$\Pr\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \Pr(A_i).$$

For $k = n$ we just say the events 'are independent'.

$D_1 = 1$
$D_2 = 1$
$D_3 = 1$

$D_1 \quad D_2 \quad D_3$

$A: D_1 = D_2$

$B: D_2 = D_3$

$C: D_1 = D_3$

$P(A) = \frac{1}{6}$

$P(B) = \frac{1}{6}$

$P(C)$

$P(A \cap B) = \frac{1}{36}$

$= P(A) \cdot P(B)$

$P(A \cap B \cap C) = \frac{1}{36}$

$\neq P(A) \cdot P(B) \cdot P(C)$

Independence

Sets of events: For a set of n events, $A_1, \ldots, A_n$, the events are *k-wise independent* if for any subset S of at most k events,

$$\Pr\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \Pr(A_i).$$

For $k = n$ we just say the events 'are independent'.

Random Variables: Two random variables X, Y are independent if for all s, t, $X = s$ and $Y = t$ are independent events. In other words:

$$\Pr(X = s \cap Y = t) = \Pr(X = s) \cdot \Pr(Y = t).$$

Application 1: Polynomial Identity Testing

Polynomial Identity Testing

Given an n-variable degree-d polynomial $p(x_1, x_2, \ldots, x_n)$, determine if the polynomial is identically zero. I.e., if $p(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n$.

Polynomial Identity Testing

Given an n-variable degree-d polynomial $p(x_1, x_2, \ldots, x_n)$, determine if the polynomial is identically zero. I.e., if $p(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n$. E.g., you are given:

$$p(x_1, x_2, \ldots, x_3) = x_3(x_1 - x_2)^3 + (x_1 + 2x_2 - x_3)^2 - x_1(x_2 + x_3)^2.$$

$$x_3 x_1^2 \cdot x_3 \cdot 2x_1 x_2 + \ldots \ldots \stackrel{?}{=} 0$$

Polynomial Identity Testing

Given an n-variable degree-d polynomial $p(x_1, x_2, \ldots, x_n)$, determine if the polynomial is identically zero. I.e., if $p(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n$. E.g., you are given:

$$p(x_1, x_2, \ldots, x_3) = x_3(x_1 - x_2)^3 + (x_1 + 2x_2 - x_3)^2 - x_1(x_2 + x_3)^2.$$

$\underline{x_1 x_3 x_2^3}$

- Can expand out all the terms and check if they cancel. But the number of terms can be as large as $\binom{n+d}{d}$ – i.e., exponential in the number of variables n and the degree d. $\approx n^d$

Polynomial Identity Testing

Given an n-variable degree-d polynomial $p(x_1, x_2, \ldots, x_n)$, determine if the polynomial is identically zero. I.e., if $p(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n$. E.g., you are given:

$$p(x_1, x_2, \ldots, x_3) = x_3(x_1 - x_2)^3 + (x_1 + 2x_2 - x_3)^2 - x_1(x_2 + x_3)^2.$$

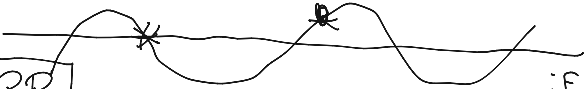
- Can expand out all the terms and check if they cancel. But the number of terms can be as large as $\binom{n+d}{d}$ – i.e., exponential in the number of variables n and the degree d.

Extremely Simple Randomized Algorithm: Just pick random values for $x_1, \ldots, x_n$ and evaluate the polynomial at these values. With high probability, if $p(x_1, \ldots, x_n) = 0$, the polynomial is identically 0!

ZPP ?
RP
BPP

$$p(5, \underline{2, \ldots, -1}) = -1(5-2)^3 + (5 + 2 \cdot 2 + 1)^2 - 5(2-1)^2 \underline{= 68.}$$

0



$\boxed{\text{CO-RP}}$

if $p \neq 0$
(may) answer incorrectly

if $p = 0$
always correctly

Polynomial Identity Testing

Given an n-variable degree-d polynomial $p(x_1, x_2, \ldots, x_n)$, determine if the polynomial is identically zero. I.e., if $p(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n$. E.g., you are given:

$$p(x_1, x_2, \ldots, x_3) = x_3(x_1 - x_2)^3 + (x_1 + 2x_2 - x_3)^2 - x_1(x_2 + x_3)^2.$$

- Can expand out all the terms and check if they cancel. But the number of terms can be as large as $\binom{n+d}{d}$ – i.e., exponential in the number of variables n and the degree d.

Extremely Simple Randomized Algorithm: Just pick random values for $x_1, \ldots, x_n$ and evaluate the polynomial at these values. With high probability, if $p(x_1, \ldots, x_n) = 0$, the polynomial is identically 0!

$$p(5, 2, \ldots, -1) = -1(5-2)^3 + (5 + 2 \cdot 2 + 1)^2 - 5(2-1)^2 = 68.$$

What style algorithm is this? BPP, ZPP, RP, something else?

co-RP

Polynomial Identity Testing Proof

Schwartz-Zippel Lemma: For any n-variable degree-d polynomial $0 \not\equiv p(x_1, \ldots, x_n)$ and any set S, if $z_1, \ldots, z_n$ are selected independently and uniformly at random from S, then $\Pr[p(z_1, \ldots, z_n) \neq 0] \geq 1 - \frac{d}{|S|}$.

$d = 10$

$S = \{0, 1, 2, 3, \ldots 100\}$

$1 - \frac{d}{|S|} = 1 - \frac{1}{10} = 90\%$

$\{0, .5, 1, 1.5 \ldots\}$

$x_1 - x_2$

Polynomial Identity Testing Proof

Schwartz-Zippel Lemma: For any n-variable degree-d polynomial $p(x_1, \ldots, x_n)$ and any set S, if $z_1, \ldots, z_n$ are selected independently and uniformly at random from S, then $\Pr[p(z_1, \ldots, z_n) \neq 0] \geq 1 - \frac{d}{|S|}$.

random vars

Proof: Via induction on the number of variables n

Base Case $n = 1$:

$p(x_1) = x_1^2 + 3x^3 + 4x^4 + \ldots$

p has at most d roots

$p(x) = 0$ for at most d vales of x

worst case, all roots are in S

probability that z_1 is a root so $p(z_1) = 0$ is $\leq \frac{d}{|S|}$

$\Pr(p(z_1) \neq 0) \geq 1 - \frac{d}{|S|}$

Polynomial Identity Testing Proof

Schwartz-Zippel Lemma: For any n-variable degree-d polynomial $p(x_1, \ldots, x_n)$ and any set S, if $z_1, \ldots, z_n$ are selected independently and uniformly at random from S, then $\Pr[p(z_1, \ldots, z_n) \neq 0] \geq 1 - \frac{d}{|S|}$.

Proof: Via induction on the number of variables n

Induction Step $n > 1$:

Polynomial Identity Testing Proof

Schwartz-Zippel Lemma: For any n-variable degree-d polynomial $p(x_1, \ldots, x_n)$ and any set S, if $z_1, \ldots, z_n$ are selected independently and uniformly at random from S, then $\Pr[p(z_1, \ldots, z_n) \neq 0] \geq 1 - \frac{d}{|S|}$.

Proof: Via induction on the number of variables n

Induction Step $n > 1$:

- Let k be the max degree of x_1 in $p(\cdots)$. Assume w.l.o.g. that $k > 0$. Write $p(x_1, \ldots, x_n) = x_1^k \cdot q(x_2, \ldots, x_n) + r(x_1, \ldots, x_n)$. E.g.,

$$x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2 x_3 = x_1^2 \cdot \underbrace{(x_2 + x_3)}_{q(\cdots)} + \underbrace{x_1 x_2 x_3 + x_2 x_3}_{r(\cdots)}.$$

$k = 2$

Polynomial Identity Testing Proof

Schwartz-Zippel Lemma: For any n-variable degree-d polynomial $p(x_1, \ldots, x_n)$ and any set S, if $z_1, \ldots, z_n$ are selected independently and uniformly at random from S, then $\Pr[p(z_1, \ldots, z_n) \neq 0] \geq 1 - \frac{d}{|S|}$.

Proof: Via induction on the number of variables n

Induction Step $n > 1$:

- Let k be the max degree of x_1 in $p(\cdots)$. Assume w.l.o.g. that $k > 0$. Write $p(x_1, \ldots, x_n) = x_1^k \cdot q(x_2, \ldots, x_n) + r(x_1, \ldots, x_n)$. E.g.,

$$x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2 x_3 = x_1^2 \cdot \underbrace{(x_2 + x_3)}_{q(\cdots)} + \underbrace{x_1 x_2 x_3 + x_2 x_3}_{r(\cdots)}.$$

- Observe: $q(\cdot)$ is non-zero, with $\boxed{n-1}$ variables and degree $d - k$.

Polynomial Identity Testing Proof

Schwartz-Zippel Lemma: For any n-variable degree-d polynomial $p(x_1, \ldots, x_n)$ and any set S, if $z_1, \ldots, z_n$ are selected independently and uniformly at random from S, then $\Pr[p(z_1, \ldots, z_n) \neq 0] \geq 1 - \frac{d}{|S|}$.

Proof: Via induction on the number of variables n

Induction Step $n > 1$:

- Let k be the max degree of x_1 in $p(\cdots)$. Assume w.l.o.g. that $k > 0$. Write $p(x_1, \ldots, x_n) = x_1^k \cdot q(x_2, \ldots, x_n) + r(x_1, \ldots, x_n)$. E.g.,

$$x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2 x_3 = x_1^2 \cdot \underbrace{(x_2 + x_3)}_{q(\cdots)} + \underbrace{x_1 x_2 x_3 + x_2 x_3}_{r(\cdots)}.$$

- Observe: $q(\cdot)$ is non-zero, with $n-1$ variables and degree $d-k$.
- So, by inductive assumption, $\Pr[q(z_2, \ldots, z_n) \neq 0] \geq 1 - \frac{d-k}{|S|}$.

Polynomial Identity Testing Proof

Schwartz-Zippel Lemma: For any n-variable degree-d polynomial $p(x_1, \ldots, x_n)$ and any set S, if $z_1, \ldots, z_n$ are selected independently and uniformly at random from S, then $\Pr[p(z_1, \ldots, z_n) \neq 0] \geq 1 - \frac{d}{|S|}$.

Proof: Via induction on the number of variables n

Induction Step $n > 1$:

- Let k be the max degree of x_1 in $p(\cdots)$. Assume w.l.o.g. that $k > 0$. Write $p(x_1, \ldots, x_n) = x_1^k \cdot q(x_2, \ldots, x_n) + r(x_1, \ldots, x_n)$. E.g.,

$$x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2 x_3 = x_1^2 \cdot \underbrace{(x_2 + x_3)}_{q(\cdots)} + \underbrace{x_1 x_2 x_3 + x_2 x_3}_{r(\cdots)}.$$

- Observe: $q(\cdot)$ is non-zero, with $n-1$ variables and degree $d-k$.
- So, by inductive assumption, $\Pr[q(z_2, \ldots, z_n) \neq 0] \geq 1 - \frac{d-k}{|S|}$.
- Assuming $q(z_2, \ldots, z_n) \neq 0$, then $p(x_1, z_2, \ldots, z_n)$ is a degree k non-zero univariate polynomial in x_1.

Assuming $q(z_2, \ldots, z_n) \neq 0$, then $p(x_1, z_2, \ldots, z_n)$ is a degree k non-zero univariate polynomial in x_1.

Example:

2+1 $x_1 \cdot 2 \cdot 1$ $2 \cdot 1$

$$p(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2 x_3 = x_1^2 \cdot \underbrace{(x_2 + x_3)}_{q(\cdots)} + \underbrace{x_1 x_2 x_3 + x_2 x_3}_{r(\cdots)}.$$

$$p(x_1, z_2, z_3) = p(x_1, 2, 1) = x_1^2 \cdot 3 + 2x_1 + 2.$$

$q(z_2 \ldots z_n)$

Assuming $q(z_2, \ldots, z_n) \neq 0$, then $p(x_1, z_2, \ldots, z_n)$ is a degree k non-zero univariate polynomial in x_1.

Example:

$$p(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2 x_3 = x_1^2 \cdot \underbrace{(x_2 + x_3)}_{q(\cdots)} + \underbrace{x_1 x_2 x_3 + x_2 x_3}_{r(\cdots)}.$$

$$p(x_1, z_2, z_3) = \underline{p(x_1, 2, 1)} = x_1^2 \cdot 3 + 2x_1 + 2.$$

- degree k poly in 1-variable

Next Step: Again applying the inductive hypothesis,

$$\Pr[\underline{p(z_1, \ldots z_n)} \neq 0 | \underline{q(z_2, \ldots, z_n) \neq} 0] \geq \underline{1 - \frac{k}{|S|}}.$$

Polynomial Identity Testing Proof

Assuming $q(z_2, \dots, z_n) \neq 0$, then $p(x_1, z_2, \dots, z_n)$ is a degree k non-zero univariate polynomial in x_1.

Example:

$$p(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2 x_3 = x_1^2 \cdot \underbrace{(x_2 + x_3)}_{q(\cdots)} + \underbrace{x_1 x_2 x_3 + x_2 x_3}_{r(\cdots)}.$$

$$p(x_1, z_2, z_3) = p(x_1, 2, 1) = x_1^2 \cdot 3 + 2x_1 + 2.$$

S^n

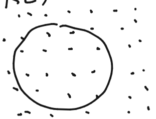
Next Step: Again applying the inductive hypothesis,

$$\Pr[p(z_1, \dots z_n) \neq 0 | q(z_2, \dots, z_n) \neq 0] \geq 1 - \frac{k}{|S|}.$$

Overall:

$$\begin{aligned} \Pr[p(z_1, \dots z_n) \neq 0] &\geq \Pr[p(z_1, \dots z_n) \neq 0 \cap q(z_2, \dots, z_n) \neq 0] \\ &= \Pr[p(\dots) \neq 0 | q(\dots) \neq 0] \cdot \Pr[q(\dots) \neq 0] \\ &\geq \left(1 - \frac{k}{|S|}\right) \cdot \left(1 - \frac{d-k}{|S|}\right) \geq 1 - \frac{d}{|S|}. \end{aligned}$$

$p(x_1, x_2) = 1 \cdot x_1^2 - x_2^2$



$\left(1 - \frac{d}{|S|} + \frac{k(d-k)}{|S|^2} \geq 1 - \frac{d}{|S|} \right.$

Polynomial Identity Testing Proof

Assuming $q(z_2, \ldots, z_n) \neq 0$, then $p(x_1, z_2, \ldots, z_n)$ is a degree k non-zero univariate polynomial in x_1.

Example:

$$p(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2 x_3 = x_1^2 \cdot \underbrace{(x_2 + x_3)}_{q(\cdots)} + \underbrace{x_1 x_2 x_3 + x_2 x_3}_{r(\cdots)}.$$

$$p(x_1, z_2, z_3) = p(x_1, 2, 1) = x_1^2 \cdot 3 + 2x_1 + 2.$$

Next Step: Again applying the inductive hypothesis,

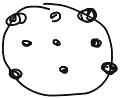
$$\Pr[p(z_1, \ldots z_n) \neq 0 | q(z_2, \ldots, z_n) \neq 0] \geq 1 - \frac{k}{|S|}.$$

Overall:

$p() = 1 - x_2^2 - x_1^2$ $\quad |S| = 2$

$$\Pr[p(z_1, \ldots z_n) \neq 0] \geq \Pr[p(z_1, \ldots z_n) \neq 0 \cap q(z_2, \ldots, z_n) \neq 0]$$

$$= \Pr[p(\ldots) \neq 0 | q(\ldots) \neq 0] \cdot \Pr[q(\ldots) \neq 0]$$



$\frac{d}{|S|} = 1$

$1 - \frac{d}{|S|} = 0$

$$\geq \left(1 - \frac{k}{|S|}\right) \cdot \left(1 - \frac{d-k}{|S|}\right) \geq 1 - \frac{d}{|S|}.$$

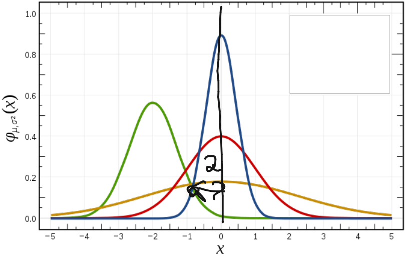
This completes the proof of Schwartz-Zippel.

Expectation and Variance Review

Expectation and Variance

Consider a random **X** variable taking values in some finite set $S \subset \mathbb{R}$. E.g., for a random dice roll, $S = \{1, 2, 3, 4, 5, 6\}$. $\{2, 4, 6, 10, 12\}$

- **Expectation:** $\mathbb{E}[\mathbf{X}] = \sum_{s \in S} \Pr(\mathbf{X} = s) \cdot s.$
- **Variance:** $\mathrm{Var}[\mathbf{X}] = \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}])^2].$



Exercise: Verify that for any scalar α, $\mathbb{E}[\alpha \cdot \mathbf{X}] = \alpha \cdot \mathbb{E}[\mathbf{X}]$ and $\mathrm{Var}[\alpha \cdot \mathbf{X}] = \alpha^2 \cdot \mathrm{Var}[\mathbf{X}]$.

Linearity of Expectation

$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ for any random variables X and Y. No matter how correlated they may be!

$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ for any random variables X and Y. No matter how correlated they may be!

Proof:

$X+Y$ when $X=s$ + $Y=t$

$$\mathbb{E}\underline{[X+Y]} = \sum_{\underline{s\in S}} \sum_{\underline{t\in T}} \Pr(\underline{X = s \cap Y = t}) \cdot \underline{(s+t)}$$

Linearity of Expectation

$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ for any random variables X and Y. No matter how correlated they may be!

Proof:

$$\mathbb{E}[X+Y] = \sum_{s\in S}\sum_{t\in T} \Pr(X = s \cap Y = t) \cdot (s + t)$$

$$= \sum_{s\in S} \underbrace{\sum_{t\in T} \Pr(X = s \cap Y = t)}_{\Pr(X=s)} \cdot s + \underbrace{\sum_{t\in T}\sum_{s\in S} \Pr(X = s \cap Y = t) \cdot t}_{\mathbb{E}[Y]}$$

$$\underbrace{\sum_{s\in S} P(X=S)\cdot S}_{\mathbb{E}(X)}$$

$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ for any random variables X and Y. No matter how correlated they may be!

Proof:

$$\begin{aligned}
\mathbb{E}[X+Y] &= \sum_{s\in S}\sum_{t\in T} \Pr(X = s \cap Y = t)\cdot(s+t) \\
&= \sum_{s\in S}\sum_{t\in T} \Pr(\underline{X = s \cap Y} = t)\cdot s + \sum_{t\in T}\sum_{s\in S} \Pr(X = \underline{s \cap Y} = t)\cdot t \\
&= \sum_{s\in S} \underline{\Pr(X = s)}\cdot s + \sum_{t\in T} \underline{\Pr(Y = t)}\cdot t
\end{aligned}$$

Linearity of Expectation

$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ for any random variables X and Y. No matter how correlated they may be!

Proof:

$$\begin{aligned}
\mathbb{E}[X+Y] &= \sum_{s \in S} \sum_{t \in T} \Pr(X = s \cap Y = t) \cdot (s+t) \\
&= \sum_{s \in S} \sum_{t \in T} \Pr(X = s \cap Y = t) \cdot s + \sum_{t \in T} \sum_{s \in S} \Pr(X = s \cap Y = t) \cdot t \\
&= \sum_{s \in S} \Pr(X = s) \cdot s + \sum_{t \in T} \Pr(Y = t) \cdot t \\
&= \mathbb{E}[X] + \mathbb{E}[Y].
\end{aligned}$$

Linearity of Expectation

$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ for any random variables X and Y. No matter how correlated they may be!

Proof:

$$
\begin{aligned}
\mathbb{E}[X + Y] &= \sum_{s \in S} \sum_{t \in T} \underline{\Pr(X = s \cap Y = t)} \cdot (s + t) \\
&= \sum_{s \in S} \sum_{t \in T} \Pr(X = s \cap Y = t) \cdot s + \sum_{t \in T} \sum_{s \in S} \Pr(X = s \cap Y = t) \cdot t \\
&= \sum_{s \in S} \Pr(X = s) \cdot s + \sum_{t \in T} \Pr(Y = t) \cdot t \\
&= \mathbb{E}[X] + \mathbb{E}[Y].
\end{aligned}
$$

Maybe the single most powerful tool in the analysis of randomized algorithms.

Linearity of Variance

$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$ when X and Y are independent.

Linearity of Variance

$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$ when X and Y are independent.

Claim 1: (exercise) $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

$\mathbb{E}(X - \mathbb{E}X)^2$

Claim 2: (exercise) $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ (i.e., X and Y are uncorrelated) when X, Y are independent.

Linearity of Variance

$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$ when X and Y are independent.

Claim 1: (exercise) $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

Claim 2: (exercise) $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ (i.e., X and Y are uncorrelated) when X, Y are independent.

Together give:

$$\text{Var}[X+Y] = \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2$$

Linearity of Variance

$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$ when X and Y are independent.

Claim 1: (exercise) $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

Claim 2: (exercise) $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ (i.e., X and Y are uncorrelated) when X, Y are independent.

Together give: $\mathbb{E}[X^2 + 2XY + Y^2]$

$$\begin{aligned}\mathrm{Var}[X+Y] &= \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2 \\ &= \underline{\mathbb{E}[X^2]} + \underline{2\mathbb{E}[XY]} + \underline{\mathbb{E}[Y^2]} - (\mathbb{E}[X] + \mathbb{E}[Y])^2\end{aligned}$$

Linearity of Variance

$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$ when X and Y are independent.

Claim 1: (exercise) $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

Claim 2: (exercise) $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ (i.e., X and Y are uncorrelated) when X, Y are independent.

Together give:

$$
\begin{aligned}
\text{Var}[X + Y] &= \mathbb{E}[(X + Y)^2] - \mathbb{E}[X + Y]^2 \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2 \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - 2\mathbb{E}[X] \cdot \mathbb{E}[Y] - \mathbb{E}[Y]^2
\end{aligned}
$$

$\text{Var}(X) + \text{Var}(Y) + 2\mathbb{E}XY - 2\mathbb{E}X\mathbb{E}Y$

0 by claim 2

Linearity of Variance

$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$ when X and Y are independent.

Claim 1: (exercise) $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

Claim 2: (exercise) $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ (i.e., X and Y are uncorrelated) when X, Y are independent.

Together give:

$$\begin{aligned}
\text{Var}[X+Y] &= \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2 \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2 \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - 2\mathbb{E}[X] \cdot \mathbb{E}[Y] - \mathbb{E}[Y]^2 \\
&= \mathbb{E}[X^2] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2
\end{aligned}$$

Linearity of Variance

$\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$ when X and Y are independent.

Claim 1: (exercise) $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

Claim 2: (exercise) $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ (i.e., X and Y are uncorrelated) when X, Y are independent.

Together give:

$$\begin{aligned}
\mathrm{Var}[X + Y] &= \mathbb{E}[(X + Y)^2] - \mathbb{E}[X + Y]^2 \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2 \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - 2\mathbb{E}[X] \cdot \mathbb{E}[Y] - \mathbb{E}[Y]^2 \\
&= \mathbb{E}[X^2] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 \\
&= \mathrm{Var}[X] + \mathrm{Var}[Y].
\end{aligned}$$

Linearity of Variance

Exercise: Verify that for random variables $X_1, \ldots, X_n$,

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i),$$

whenever the variables are 2-wise independent (also called pairwise independent).

Application 2: Quicksort with Random Pivots

Quicksort

`Quicksort`(X): where $X = (x_1, \ldots, x_n)$ is a list of numbers.

1. If X is empty: return X.
2. Else: select pivot p uniformly at random from $\{1, \ldots, n\}$.
3. Let $X_{lo} = \{i \in X : x_i < x_p\}$ and $X_{hi} = \{i \in X : x_i \geq x_p\}$ (requires $n - 1$ comparisons with x_p to determine).
4. Return the concatenation of the lists [`Quicksort`(X_{lo}), (x_p), `Quicksort`(X_{hi})].

Quicksort

`Quicksort`(X): where $X = (x_1, \ldots, x_n)$ is a list of numbers.

1. If X is empty: return X.
2. Else: select pivot p uniformly at random from $\{1, \ldots, n\}$.
3. Let $X_{lo} = \{i \in X : x_i < x_p\}$ and $X_{hi} = \{i \in X : x_i \geq x_p\}$ (requires $n - 1$ comparisons with x_p to determine).
4. Return the concatenation of the lists [`Quicksort`(X_{lo}), (x_p), `Quicksort`(X_{hi})].

4	5	2	8	1	3	6	9	7	0

Quicksort

`Quicksort`(X): where $X = (x_1, \ldots, x_n)$ is a list of numbers.

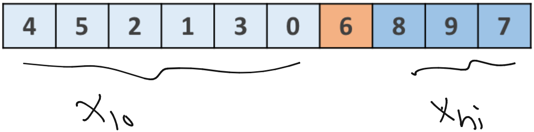
1. If X is empty: return X.
2. Else: select pivot p uniformly at random from $\{1, \ldots, n\}$.
3. Let $X_{lo} = \{i \in X : x_i < x_p\}$ and $X_{hi} = \{i \in X : x_i \geq x_p\}$ (requires $n - 1$ comparisons with x_p to determine).
4. Return the concatenation of the lists [`Quicksort`(X_{lo}), (x_p), `Quicksort`(X_{hi})].

4	5	2	8	1	3	6	9	7	0

Quicksort

`Quicksort`(X): where $X = (x_1, \ldots, x_n)$ is a list of numbers.

1. If X is empty: return X.
2. Else: select pivot p uniformly at random from $\{1, \ldots, n\}$.
3. Let $X_{lo} = \{i \in X : x_i < x_p\}$ and $X_{hi} = \{i \in X : x_i \geq x_p\}$ (requires $n - 1$ comparisons with x_p to determine).
4. Return the concatenation of the lists [`Quicksort`(X_{lo}), (x_p), `Quicksort`(X_{hi})].



Quicksort

`Quicksort`(X): where $X = (x_1, \ldots, x_n)$ is a list of numbers.

1. If X is empty: return X.
2. Else: select pivot p uniformly at random from $\{1, \ldots, n\}$.
3. Let $X_{lo} = \{i \in X : x_i < x_p\}$ and $X_{hi} = \{i \in X : x_i \geq x_p\}$ (requires $n-1$ comparisons with x_p to determine).
4. Return the concatenation of the lists [`Quicksort`(X_{lo}), (x_p), `Quicksort`(X_{hi})].

0	1	2	3	4	5	6	7	8	9

$\mathbb{E}[T] = \underline{O(n \log n)}$

Quicksort

`Quicksort`(X): where $X = (x_1, \ldots, x_n)$ is a list of numbers.

1. If X is empty: return X.
2. Else: select pivot p uniformly at random from $\{1, \ldots, n\}$.
3. Let $X_{lo} = \{i \in X : x_i < x_p\}$ and $X_{hi} = \{i \in X : x_i \geq x_p\}$ (requires $n - 1$ comparisons with x_p to determine).
4. Return the concatenation of the lists [`Quicksort`(X_{lo}), (x_p), `Quicksort`(X_{hi})].

0	1	2	3	4	5	6	7	8	9

What is the worst case running time of this algorithm?

$O(n^2)$

Randomized Quicksort Analysis

Theorem: Let T be the number of comparisions performed by `Quicksort`(X). Then $\mathbb{E}[T] = O(n \log n)$.

Theorem: Let T be the number of comparisions performed by `Quicksort`(X). Then $\mathbb{E}[\mathrm{T}] = O(n \log n)$.

- For any $i, j \in [n]$ with $i < j$, let $\mathrm{I}_{ij} = 1$ if x_i, x_j are compared at some point during the algorithm, and $\mathrm{I}_{ij} = 0$ if they are not. An indicator random variable.

Randomized Quicksort Analysis

Theorem: Let T be the number of comparisions performed by `Quicksort`(X). Then $\mathbb{E}[\mathsf{T}] = O(n \log n)$.

- For any $i, j \in [n]$ with $i < j$, let $\mathsf{I}_{ij} = 1$ if x_i, x_j are compared at some point during the algorithm, and $\mathsf{I}_{ij} = 0$ if they are not. An indicator random variable.
- We can write $\underline{\mathsf{T}} = \underline{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathsf{I}_{ij}}$.

Randomized Quicksort Analysis

Theorem: Let $\mathbf{T}$ be the number of comparisions performed by `Quicksort`(X). Then $\mathbb{E}[\mathbf{T}] = O(n \log n)$.

- For any $i, j \in [n]$ with $i < j$, let $\mathbf{I}_{ij} = 1$ if x_i, x_j are compared at some point during the algorithm, and $\mathbf{I}_{ij} = 0$ if they are not. An indicator random variable. (are these independent?
- We can write $\mathbf{T} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbf{I}_{ij}$. Thus, via linearity of expectation

$$\mathbb{E}[\mathbf{T}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbb{E}[\mathbf{I}_{ij}]$$

Randomized Quicksort Analysis

Theorem: Let $\mathbf{T}$ be the number of comparisions performed by $\texttt{Quicksort}(X)$. Then $\mathbb{E}[\mathbf{T}] = O(n \log n)$.

- For any $i, j \in [n]$ with $i < j$, let $\mathbf{I}_{ij} = 1$ if x_i, x_j are compared at some point during the algorithm, and $\mathbf{I}_{ij} = 0$ if they are not. An indicator random variable.
- We can write $\mathbf{T} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbf{I}_{ij}$. Thus, via linearity of expectation

$$\underline{\mathbb{E}[\mathbf{T}]} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \underline{\mathbb{E}[\mathbf{I}_{ij}]} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \underline{\Pr[x_i, x_j \text{ are compared}]}$$

Theorem: Let T be the number of comparisions performed by `Quicksort`(X). Then $\mathbb{E}[\mathsf{T}] = O(n \log n)$.

- For any $i, j \in [n]$ with $i < j$, let $\mathbf{I}_{ij} = 1$ if x_i, x_j are compared at some point during the algorithm, and $\mathbf{I}_{ij} = 0$ if they are not. An indicator random variable.
- We can write $\mathsf{T} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbf{I}_{ij}$. Thus, via linearity of expectation

$$\mathbb{E}[\mathsf{T}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbb{E}[\mathbf{I}_{ij}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[x_i, x_j \text{ are compared}]$$

So we need to upper bound $\Pr[x_i, x_j \text{ are compared}]$.

Randomized Quicksort Analysis

Upper bounding $\Pr[x_i, x_j \text{ are compared}]$:

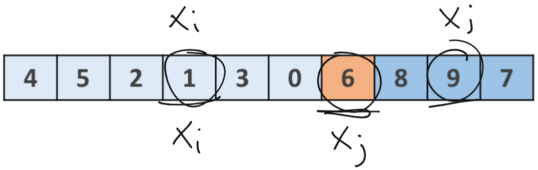
Randomized Quicksort Analysis

Upper bounding $\Pr[x_i, x_j \text{ are compared}]$: $x_i < x_j$

- Assume without loss of generality that $x_1 \leq x_2 \leq \ldots \leq x_n$. This is just 'renaming' the elements of our list. Also recall that $i < j$.

Upper bounding $\Pr[x_i, x_j$ **are compared]:**

- Assume without loss of generality that $x_1 \leq x_2 \leq \ldots \leq x_n$. This is just 'renaming' the elements of our list. Also recall that $i < j$.
- At exactly one step of the recursion, x_i, x_j will be 'split up' with one landing in X_{hi} and the other landing in X_{lo}, or one being chosen as the pivot. x_i, x_j are only ever compared in this later case – if one is chosen as the pivot when they are split up.



Randomized Quicksort Analysis

Upper bounding $\Pr[x_i, x_j$ are compared$]$:

- Assume without loss of generality that $x_1 \leq x_2 \leq \ldots \leq x_n$. This is just 'renaming' the elements of our list. Also recall that $i < j$.
- At exactly one step of the recursion, x_i, x_j will be 'split up' with one landing in X_{hi} and the other landing in X_{lo}, or one being chosen as the pivot. x_i, x_j are only ever compared in this later case – if one is chosen as the pivot when they are split up.
- The split occurs when some element between x_i and x_j is chosen as the pivot. The possible elements are $\underbrace{x_i, x_{i+1}, \ldots, x_j}_{j-i+1}$.

4	5	2	1	3	0	6	8	9	7

$$\Pr(x_i + x_j \text{ are compared}) = \frac{2}{j-i+1}$$

Upper bounding $\Pr[x_i, x_j$ are compared]:

- Assume without loss of generality that $x_1 \leq x_2 \leq \ldots \leq x_n$. This is just 'renaming' the elements of our list. Also recall that $i < j$.
- At exactly one step of the recursion, x_i, x_j will be 'split up' with one landing in X_{hi} and the other landing in X_{lo}, or one being chosen as the pivot. x_i, x_j are only ever compared in this later case – if one is chosen as the pivot when they are split up.
- The split occurs when some element between x_i and x_j is chosen as the pivot. The possible elements are $x_i, x_{i+1}, \ldots, x_j$.

4	5	2	1	3	0	6	8	9	7

- $\Pr[x_i, x_j$ are compared] is equal to the probability that either x_i or x_j are chosen as the splitting pivot from this list. Thus, $\Pr[x_i, x_j$ are compared] $= \frac{2}{j-i+1}$

Randomized Quicksort Analysis

So Far: Expected number of comparisons is given as:

$$\mathbb{E}[\mathsf{T}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[x_i, \underline{x_j \text{ are compared}}].$$

And we computed $Pr[x_i, x_j$ are compared$] = \underline{\frac{2}{j-i+1}}$.

So Far: Expected number of comparisons is given as:

$$\mathbb{E}[T] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[x_i, x_j \text{ are compared}].$$

And we computed $Pr[x_i, x_j \text{ are compared}] = \frac{2}{j-i+1}$. Plugging in:

$$\mathbb{E}[T] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

Randomized Quicksort Analysis

So Far: Expected number of comparisons is given as:

$$\mathbb{E}[\mathrm{T}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[x_i, x_j \text{ are compared}].$$

And we computed $Pr[x_i, x_j \text{ are compared}] = \frac{2}{j-i+1}$. Plugging in:

$$\mathbb{E}[\mathrm{T}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = \sum_{i=1}^{n-1} \sum_{\underbrace{k=2}_{j-i+1}}^{n-i+1} \frac{2}{k}$$

So Far: Expected number of comparisons is given as:

$$\mathbb{E}[\mathrm{T}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[x_i, x_j \text{ are compared}].$$

And we computed $Pr[x_i, x_j \text{ are compared}] = \frac{2}{j-i+1}$. Plugging in:

$$\mathbb{E}[\mathrm{T}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = \sum_{i=1}^{n-1} \sum_{k=2}^{n-i+1} \frac{2}{k}$$

$$\leq \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} \leq 2 \cdot (n-1) \cdot \sum_{k=1}^{n} \frac{1}{k}$$

So Far: Expected number of comparisons is given as:

$$\mathbb{E}[\mathrm{T}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[x_i, x_j \text{ are compared}].$$

And we computed $Pr[x_i, x_j \text{ are compared}] = \frac{2}{j-i+1}$. Plugging in:

$$\mathbb{E}[\mathrm{T}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = \sum_{i=1}^{n-1} \sum_{k=2}^{n-i+1} \frac{2}{k}$$

$$\leq \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} \leq 2 \cdot (n-1) \cdot \underline{\sum_{k=1}^{n} \frac{1}{k}} = 2n \cdot \underline{H_n} = O(n \log n).$$