COMPSCI 614: Randomized Algorithms with Applications to Data Science

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University of Massachusetts Amherst. Spring 2024.

Lecture 19

Logistics

- I will send responses to project progress reports soon.

Summary

Last Week: Start on Markov Chains.

- Start on Markov chains and their analysis
- Markov chain based algorithms for satisfiability: $\approx n^2$ time for 2-SAT, and $\approx (4/3)^n$ for 3-SAT.

Today: Markov Chains Continued

- The gambler's ruin problem.
- Aperiodicity and stationary distribution of a Markov chain.
- The fundamental theorem of Markov chains.

Markov Chain Review

- A discrete time stochastic process is a Markov chain if is it memoryless:

$$\Pr(X_t = a_t | X_{t-1} = a_{t-1}, \ldots, X_0 = a_0) = \Pr(X_t = a_t | X_{t-1} = a_{t-1})$$

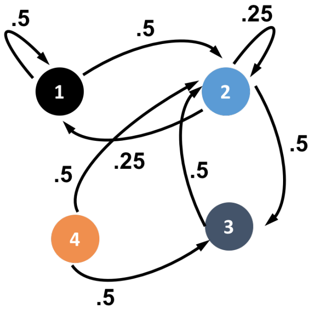
- If each X_t can take m possible values, the Markov chain is specified by the transition matrix $P \in [0,1]^{m \times m}$ with

$$P_{i,j} = \Pr(X_{t+1} = j | X_t = i).$$

- Let $q_t \in [0,1]^{1 \times m}$ be the distribution of X_i. Then $q_{t+1} = q_t P$.

$$\underbrace{\begin{bmatrix} .5 & .5 & 0 & 0 \end{bmatrix}}_{\mathbf{q_1}} \underbrace{\begin{bmatrix} .5 & .5 & 0 & 0 \\ .25 & .25 & .5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & .5 & .5 & 0 \end{bmatrix}}_{\mathbf{P}} = \underbrace{\begin{bmatrix} .375 & .375 & .25 & 0 \end{bmatrix}}_{\mathbf{q_2}}$$

Often viewed as an underlying state transition graph. Nodes correspond to possible values that each X_t can take.



P

.5	.5	0	0
.25	.25	.5	0
0	1	0	0
0	.5	.5	0

The Markov chain is irreducible if the underlying graph consists of single strongly connected component.

Gambler’s Ruin

Gambler's Ruin

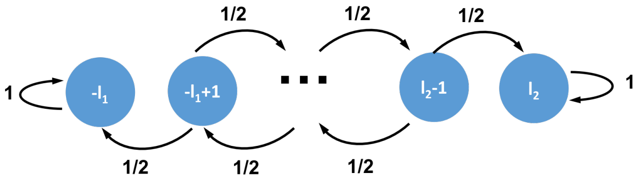
- You and 'a friend' repeatedly toss a fair coin. If it hits heads, you give your friend \$1. If it hits tails, they give you \$1.
- You start with $\$\ell_1$ and your friend starts with $\$\ell_2$. When either of you runs out of money the game terminates.
- What is the probability that you win $\$\ell_2$?

Gambler's Ruin Markov Chain

Let $X_0, X_1, \ldots$ be the Markov chain where X_t is your profit at step t. $X_0 = 0$ and:

$$P_{-\ell_1,-\ell_1} = P_{\ell_2,\ell_2} = 1$$

$$P_{i,i+1} = P_{i,i-1} = 1/2 \text{ for } -\ell_1 < i < \ell_2$$



- ℓ_1 and ℓ_2 are absorbing states.
- All i with $-\ell_1 < i < \ell_2$ are transient states. I.e., $\Pr[X_{t'} = i \text{ for some } t' > t \mid X_t = i] < 1$.

Observe that this Markov chain is also a Martingale since $\mathbb{E}[X_{t+1}|X_t] = X_t$.

Gambler's Ruin Analysis

Let $X_0, X_1, \ldots$ be the Markov chain where X_t is your profit at step t. $X_0 = 0$ and:

$$P_{-\ell_1,-\ell_1} = P_{\ell_2,\ell_2} = 1$$
$$P_{i,i+1} = P_{i,i-1} = 1/2 \text{ for } -\ell_1 < i < \ell_2$$

We want to compute $q = \lim_{t\to\infty} \Pr[X_t = \ell_2]$.

By linearity of expectation, for any i, $\mathbb{E}[X_i] = 0$. Further, for $q = \lim_{t\to\infty} \Pr[X_t = \ell_2]$, since $-\ell_1, \ell_2$ are the only non-transient states,

$$\lim_{t\to\infty} \mathbb{E}[X_t] = \ell_2 q + -\ell_1(1-q) = 0.$$

Solving for q, we have $q = \frac{\ell_1}{\ell_1+\ell_2}$.

Gambler's Ruin Thought Exercise

What if you always walk away as soon as you win just $1. Then what is your probability of winning, and what are your expected winnings?

Stationary Distributions

Stationary Distribution

A stationary distribution of a Markov chain with transition matrix $P \in [0,1]^{m \times m}$ is a distribution $\pi \in [0,1]^m$ such that $\pi = \pi P$.

I.e. if $X_t \sim \pi$, then $X_{t+1} \sim \pi P = \pi$.

$$\overset{\boldsymbol{\pi}}{\begin{bmatrix} .4 & .4 & .2 & 0 \end{bmatrix}} \overset{\mathbf{P}}{\begin{bmatrix} .5 & .5 & 0 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & .5 & .5 & 0 \end{bmatrix}} = \overset{\boldsymbol{\pi}}{\begin{bmatrix} .4 & .4 & .2 & 0 \end{bmatrix}}$$

Think-pair-share: Do all Markov chains have a stationary distribution?

Claim (Existence of Stationary Distribution)

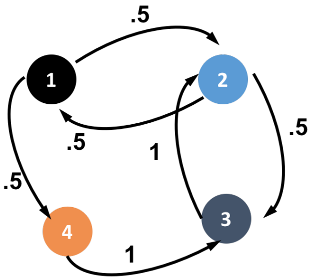
Any Markov chain with a finite state space, and transition matrix $P \in [0,1]^{m \times m}$ has a stationary distribution $\pi \in [0,1]^m$ with $\pi = \pi P$.

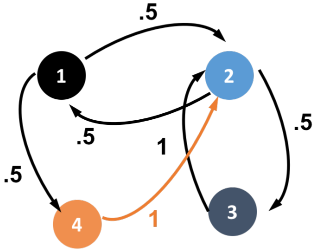
Follows from the Brouwer fixed point theorem: for any continuous function $f : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is a compact convex set, there is some x such that $f(x) = x$.

Periodicity

The periodicity of a state i is defined as:

$$T = \gcd\{t > 0 : \Pr(X_t = i \mid X_0 = i) > 0\}.$$





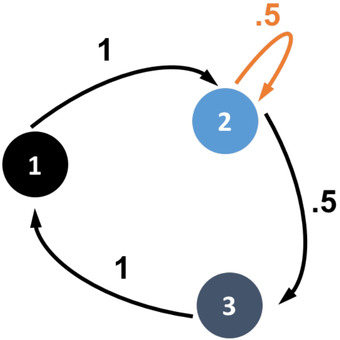
The state is aperiodic if it has periodicity $T = 1$.

A Markov chain is aperiodic if all states are aperiodic.

Periodicity

Claim

If a Markov chain is irreducible, and has at least one self-loop, then it is aperiodic.



Fundamental Theorem

Theorem (The Fundamental Theorem of Markov Chains)

Let $X_0, X_1, \ldots$ *be a Markov chain with a finite state space and transition matrix* $P \in [0,1]^{m \times m}$*. If the chain is both irreducible and aperiodic,*

1. *There exists a unique stationary distribution* $\pi \in [0,1]^m$ *with* $\pi = \pi P$*.*
2. *For any states* i, j, $\lim_{t \to \infty} \Pr[X_t = i | X_0 = j] = \pi(i)$*. I.e., for any initial distribution* q_0, $\lim_{t \to \infty} q_t = \lim_{t \to \infty} q_0 P^t = \pi$*.*
3. $\pi(i) = \frac{1}{\mathbb{E}[\min(t : X_t = i) | X_0 = i]}$*. I.e.,* $\pi(i)$ *is the inverse of the average expected return time from state* i *back to* i*.*

In the limit, the probability of being at any state i is independent of the starting state.

Stationary Distribution Example 1

Shuffling Markov Chain: Given a pack of c cards. At each step draw two random cards, swap them and repeat.

- What is the state space of this chain?
- What is the transition probability $P_{i,j}$? How does it compare to $P_{j,i}$?
- This Markov chain is symmetric and thus its stationary distribution is uniform, $\pi(i) = \frac{1}{c!}$.

Letting $m = c!$ denote the size of the state space,

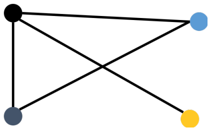
$$\pi P_{:,i} = \sum_j \pi(j) P_{j,i} = \sum_j \pi(j) P_{i,j} = \frac{1}{m} \sum_j P_{i,j} = \frac{1}{m} = \pi(i).$$

Once we have exhibited a stationary distribution, we know that it is unique and that the chain converges to it in the limit!

Stationary Distribution Example 2

Random Walk on an Undirected Graph: Consider a random walk on an undirected graph. If it is at node i at step t, then it moves to any of i's neighbors at step $t+1$ with probability $\frac{1}{d_i}$.

- What is the state space of this chain?
- What is the transition probability $P_{i,j}$?
- Is this chain aperiodic?
- If the graph is not bipartite, then there is at least one odd cycle, making the chain aperiodic.



Stationary Distribution Example 2

Random Walk on an Undirected Graph: Consider a random walk on an undirected graph. If it is at node i at step t, then it moves to any of i's neighbors at step $t+1$ with probability $\frac{1}{d_i}$.

Claim: When the graph is not bipartite, the unique stationary distribution of this Markov chain is given by $\pi(i) = \frac{d_i}{2|E|}$.

$$\pi P_{:,i} = \sum_j \pi(j) P_{j,i} = \sum_j \frac{d_j}{2|E|} \cdot \frac{1}{d_j} = \sum_j \frac{1}{2|E|} = \frac{d_i}{2|E|} = \pi(i).$$

I.e., the probability of being at a given node i is dependent only on the node's degree, not on the structure of the graph in any other way.