COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2023.

Lecture 8

Summary

Last Class:

- Bloom filter analysis and optimization of parameters.

$$k = \frac{\ln 2 \cdot m}{n}$$

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- Bloom filter analysis and optimization of parameters.

This Class:

- Streaming algorithms and distinct elements estimation via hashing.
- Analysis of the distinct elements algorithm.
- The median trick for boosting success probability.
- Sketch of the ideas behind practical algorithms for distinct elements estimation.

Stream Processing: Have a massive dataset X with n items $x_1, x_2, \ldots, x_n$ that arrive in a continuous stream. Not nearly enough space to store all the items (in a single location).

- Still want to analyze and learn from this data.
- Typically must compress the data on the fly, storing a data structure from which you can still learn useful information.
- Often the compression is randomized. E.g., bloom filters.
- Compared to traditional algorithm design, which focuses on minimizing runtime, the big question here is how much space is needed to answer queries of interest.

Some Examples

- **Sensor data:** images from telescopes (30 terabytes per night from the Vera C. Rubin Observatory), readings from seismometer arrays monitoring and predicting earthquake activity, traffic cameras and travel time sensors (Smart Cities), electrical grid monitoring.
- **Internet Traffic**: 8.5 billion Google searches, billions of ad-clicks and other logs from instrumented webpages, IPs routed by network switches, ...
- **Datasets in Machine Learning:** When training e.g. a neural network on a large dataset (ImageNet with 14 million images or LLaMA-2 on trillions of tokens of text), the data is typically processed in a stream due to storage limitations.

Distinct Elements

Distinct Elements (Count-Distinct) Problem: Given a stream $x_1, \ldots, x_n$, estimate the number of distinct elements in the stream. E.g.,

$$1, 5, 7, 5, 2, 1 \rightarrow 4 \text{ distinct elements}$$

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n=0

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when x arrives

n=n+1

Applications:

- Distinct IP addresses clicking on an ad or visiting a site.
- Distinct values in a database column (for estimating sizes of joins and group bys).

count discount

- Number of distinct search engine queries.
- Counting distinct motifs in large DNA sequences.

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$(1-\varepsilon)d \leq \tilde{d} \leq (1+\varepsilon)d$

d = 0
when x arrives
if x ∈ BF, do nothing
else d = d+1; add x to BF

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Google Sawzall, Facebook Presto, Apache Drill, Twitter Algebird

Hashing for Distinct Elements

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Min-Hashing for Distinct Elements (variant of Flajolet-Martin):

- Let $\mathbf{h} : U \to [0, 1]$ be a random hash function (with a real valued output)

$h(x) = .7326$

- $s := 1$
- For $i = 1, \ldots, n$
  - $s := \min(s, \mathbf{h}(x_i))$
- Return $\tilde{d} = \frac{1}{s} - 1$

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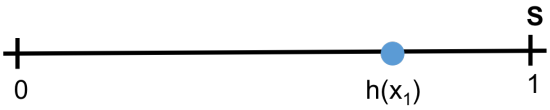

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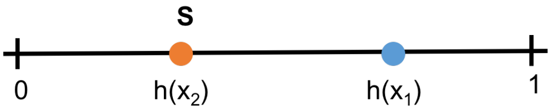

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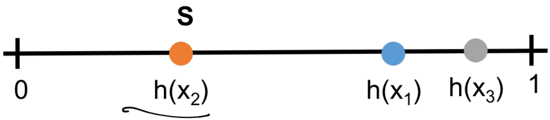

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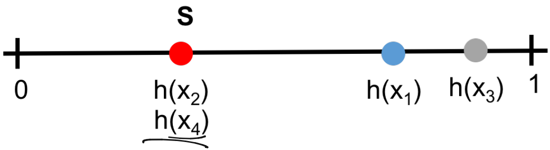
a z b z
x_1 x_2 x_3 x_4



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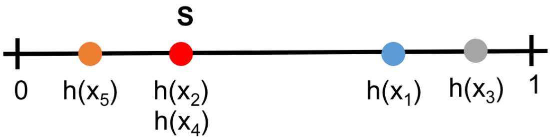

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d = distinct elements
n = elements

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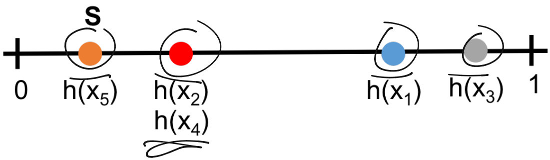
s = min of d random values.

n = 0
s = 1



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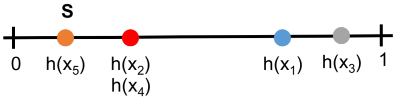

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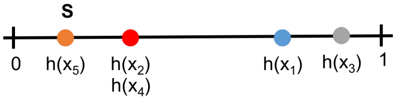

- After all items are processed, s is the minimum of d points chosen uniformly at random on $[0, 1]$. Where $d = \#$ distinct elements.

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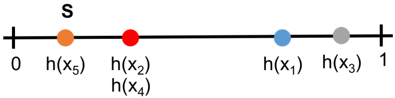

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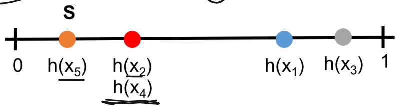
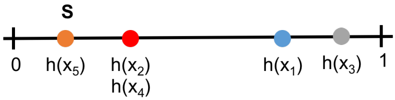
$d = 4$ $\quad h(x_1) = \frac{1}{4}$ $\quad \frac{1}{1/4} - 1 = 3$



- After all items are processed, s is the minimum of d points chosen uniformly at random on $[0,1]$. Where $d = \#$ distinct elements.
- Intuition: The larger d is, the smaller we expect s to be.
- Same idea as Flajolet-Martin algorithm and HyperLogLog, except they use discrete hash functions.

Performance in Expectation

s is the minimum of $\boxed{d \text{ points}}$ chosen uniformly at random on $[0, 1]$.
Where $d = \#$ distinct elements.



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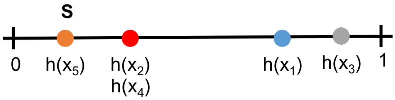

$$\mathbb{E}[s] = \frac{1}{d+1}$$

$d = 1$
$s = \frac{1}{2}$

$d = 2$
$s = \frac{1}{3}$

0 $\quad \frac{1}{3} \quad$ 2/3 $\quad$ 1

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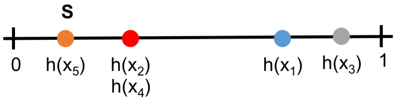

$$\mathbb{E}[s] = \frac{1}{d+1} \text{ (using } \mathbb{E}(s) = \int_0^\infty \Pr(s > x)dx) + \text{calculus)}$$

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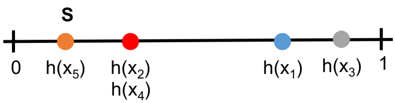

$$\mathbb{E}[\mathbf{s}] = \frac{1}{d+1} \text{ (using } \mathbb{E}(\mathbf{s}) = \int_0^\infty \Pr(\mathbf{s} > x)dx) + \text{calculus)}$$

$Pr(s > t) = 0$ when $t > 1$

- So our estimate $\widehat{\mathbf{d}} = \frac{1}{\mathbf{s}} - 1$ is correct if **s** exactly equals its expectation.

$$\frac{1}{\frac{1}{d+1}} - 1 = d+1-1 = d$$

Performance in Expectation

$\mathbf{s}$ is the minimum of d points chosen uniformly at random on $[0,1]$.
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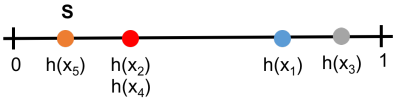
$\underline{C}$ collision



$\hat{n} = \frac{\binom{w}{2}}{C}$

$\mathbb{E}\hat{n}$

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- So our estimate $\widehat{\mathbf{d}} = \frac{1}{\mathbf{s}} - 1$ is correct if $\mathbf{s}$ exactly equals its expectation. Does this mean $\mathbb{E}[\widehat{\mathbf{d}}] = d$? ✓

$\mathbb{E}[s] = \frac{1}{d+1}$ $\qquad$ $\mathbb{E}\hat{d} = \left[\mathbb{E}\frac{1}{s}\right] - 1$

$\mathbb{E}\frac{1}{s} = d+1$ $\qquad$ $\cancel{\frac{1}{\mathbb{E}s} = d+1}$

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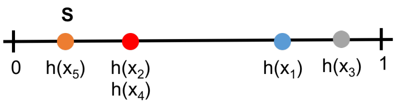

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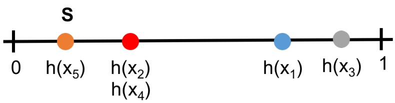

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- **Approximation is robust:** if $|\mathbf{s} - \mathbb{E}[\mathbf{s}]| \leq \epsilon \cdot \mathbb{E}[\mathbf{s}]$ for any $\epsilon \in (0, 1/2)$ and a small constant $c \leq 4$:

$$(1 - c\epsilon)d \leq \widehat{\mathbf{d}} \leq (1 + c\epsilon)d$$

Initial Concentration Bound

So question is how well s concentrates around its mean.

$$\mathbb{E}[\mathsf{s}] = \frac{1}{d+1}$$

s: minimum of d distinct hashes chosen randomly over $[0,1]$, computed by hashing algorithm. $\widehat{\mathsf{d}} = \frac{1}{\mathsf{s}} - 1$: estimate of # distinct elements d.

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$$\mathbb{E}[\mathbf{s}] = \frac{1}{d+1} \text{ and } \mathrm{Var}[\mathbf{s}] \leq \frac{1}{(d+1)^2} \text{ (\textit{also via calculus})}.$$

Chebyshevs

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Chebyshev's Inequality:

$$\Pr\left[|\mathsf{s} - \mathbb{E}[\mathsf{s}]| \geq \epsilon\mathbb{E}[\mathsf{s}]\right] \leq \frac{\mathrm{Var}[\mathsf{s}]}{(\epsilon\mathbb{E}[\mathsf{s}])^2} \qquad .$$

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$\epsilon \in (0,1)$

$$\Pr\left[|s - \mathbb{E}[s]| \geq \epsilon\mathbb{E}[s]\right] \leq \frac{\mathrm{Var}[s]}{(\epsilon\mathbb{E}[s])^2} = \frac{1}{\epsilon^2}.$$

$\epsilon = .01$

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(higher moment? Change hash distribution?)

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Bound is vacuous for any $\epsilon < 1$.

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Bound is vacuous for any $\epsilon < 1$. How can we improve accuracy?

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Leverage the law of large numbers: improve accuracy via repeated independent trials.

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  - For j=1,...,k, $\mathbf{s}_j := \min(\mathbf{s}_j, \mathbf{h}_j(x_i))$
- $\mathbf{s} := \frac{1}{k}\sum_{j=1}^{k} \mathbf{s}_j$
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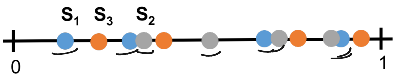
    - For j=1,...,k, $\mathbf{s}_j := \min(\mathbf{s}_j, \mathbf{h}_j(x_i))$
- $\mathbf{s} := \frac{1}{k}\sum_{j=1}^{k} \mathbf{s}_j$
- Return $\underline{\widehat{\mathbf{d}} = \frac{1}{\mathbf{s}}} - 1$

$d = 4$



Analysis

$\mathsf{s} = \frac{1}{k}\sum_{j=1}^{k} \mathsf{s}_j$. Have already shown that for $j = 1, \ldots, k$:

$$\mathbb{E}[\mathsf{s}_j] = \frac{1}{d+1}$$

$$\mathrm{Var}[\mathsf{s}_j] \leq \frac{1}{(d+1)^2}$$

$$\mathbb{E}[s] = \frac{1}{d+1}$$

Bernstein?

$$\mathrm{Var}(s) = \frac{1}{k(d+1)^2}$$

$$\mathrm{Var}(s) = \frac{1}{k^2}\mathrm{Var}\left(\sum_{j=1}^{k} s_j\right) = \frac{1}{k^2} \cdot k \cdot \frac{1}{(d+1)^2} = \frac{1}{k(d+1)^2}$$

s_j: minimum of d distinct hashes chosen randomly over $[0,1]$. $\mathsf{s} = \frac{1}{k}\sum_{j=1}^{k} \mathsf{s}_j$.
$\widehat{\mathsf{d}} = \frac{1}{\mathsf{s}} - 1$: estimate of # distinct elements d.

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Chebyshev Inequality:

$$\Pr\left[|\mathsf{s} - \mathbb{E}[\mathsf{s}]| \geq \epsilon \mathbb{E}[\mathsf{s}]\right] \leq \frac{\mathrm{Var}[\mathsf{s}]}{(\epsilon \mathbb{E}[\mathsf{s}])^2}$$

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Chebyshev Inequality:

$\epsilon \in (0,1)$

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Chebyshev Inequality:

$\epsilon = .01$

$\delta = .05$

$$\Pr\left[|s - \mathbb{E}[s]| \geq \epsilon \mathbb{E}[s]\right] \leq \frac{\mathrm{Var}[s]}{(\epsilon \mathbb{E}[s])^2} = \frac{\mathbb{E}[s]^2/k}{\epsilon^2 \mathbb{E}[s]^2} = \frac{1}{k \cdot \epsilon^2} \leq \delta$$

$\frac{1}{\epsilon^2 \delta} \leq k$

How should we set k if we want an error with probability at most δ?

$k = \frac{1}{\epsilon^2 \delta}$

s_j: minimum of d distinct hashes chosen randomly over $[0, 1]$. $s = \frac{1}{k}\sum_{j=1}^{k} s_j$.
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$\boxed{k = \frac{1}{\epsilon^2 \cdot \delta}.}$

> $\mathbf{s}_j$: minimum of d distinct hashes chosen randomly over $[0, 1]$. $\mathbf{s} = \frac{1}{k}\sum_{j=1}^{k} \mathbf{s}_j$.
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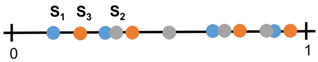
Space Complexity

Hashing for Distinct Elements:

- Let $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_k : U \to [0,1]$ be random hash functions
- $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k := 1$
- For $i = 1, \ldots, n$
  - For j=1,..., k, $\mathbf{s}_j := \min(\mathbf{s}_j, \mathbf{h}_j(x_i))$
- $\mathbf{s} := \frac{1}{k} \sum_{j=1}^{k} \mathbf{s}_j$
- Return $\widehat{\mathbf{d}} = \frac{1}{\mathbf{s}} - 1$



- Setting $k = \frac{1}{\epsilon^2 \cdot \delta}$, algorithm returns $\widehat{\mathbf{d}}$ with $|d - \widehat{\mathbf{d}}| \leq 4\epsilon \cdot d$ with probability at least $1 - \delta$.

$|S - \mathbb{E} S| \leq \epsilon \mathbb{E} S$

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- Let $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_k : U \rightarrow [0, 1]$ be random hash functions
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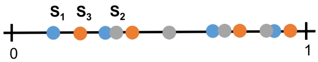

- Setting $k = \frac{1}{\epsilon^2 \cdot \delta}$, algorithm returns $\widehat{\mathbf{d}}$ with $|d - \widehat{\mathbf{d}}| \leq 4\epsilon \cdot d$ with probability at least $1 - \delta$.
- Space complexity is $k = \frac{1}{\epsilon^2 \cdot \delta}$ real numbers $s_1, \ldots, s_k$.

Hashing for Distinct Elements:

ϵ

$\delta = \frac{1}{5}$

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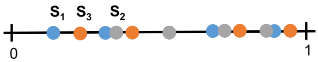

$\delta = .0001$

- Setting $k = \frac{1}{\epsilon^2 \cdot \delta}$, algorithm returns $\widehat{\mathbf{d}}$ with $|d - \widehat{\mathbf{d}}| \leq 4\epsilon \cdot d$ with probability at least $1 - \delta$.

$\frac{1}{.05} = 20$

- Space complexity is $k = \frac{1}{\epsilon^2 \cdot \delta}$ real numbers $s_1, \ldots, s_k$.
- ~~$\delta = 5\%$~~ failure rate gives a factor 20 overhead in space complexity.

Improved Failure Rate

How can we improve our dependence on the failure rate δ?

$$\delta = \frac{1}{2^{30}} \qquad \boxed{\frac{\log(1/\delta)}{\varepsilon^2}} \quad \text{vs} \quad \frac{1}{\delta \varepsilon^2}$$

Improved Failure Rate

How can we improve our dependence on the failure rate δ?

The median trick: Run $t = O(\log 1/\delta)$ trials each with failure probability $\delta' = 1/5$ – each using $k = \frac{1}{\delta' \epsilon^2} = \frac{5}{\epsilon^2}$ hash functions.

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good with at least 80% prob.

- Letting $\widehat{\mathrm{d}}_1, \ldots, \widehat{\mathrm{d}}_t$ be the outcomes of the t trials, return $\widehat{\mathrm{d}} = median(\widehat{\mathrm{d}}_1, \ldots, \widehat{\mathrm{d}}_t)$.

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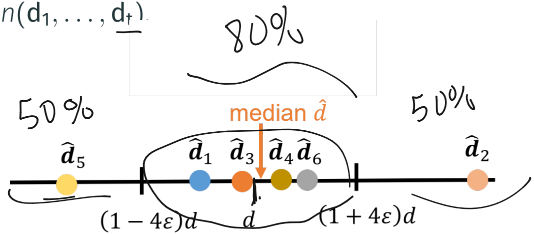

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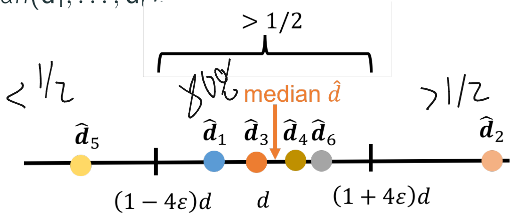

- If $> 1/2$ of trials fall in $[(1-4\epsilon)d, (1+4\epsilon)d]$, then the median will.

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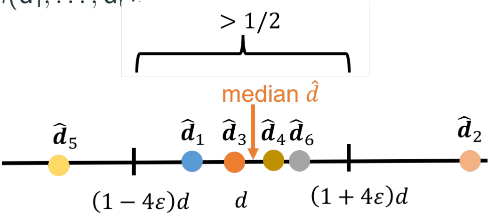

- If $> 1/2$ of trials fall in $[(1-4\epsilon)d, (1+4\epsilon)d]$, then the median will.
- Have $< 1/2$ of trials on both the left and right.

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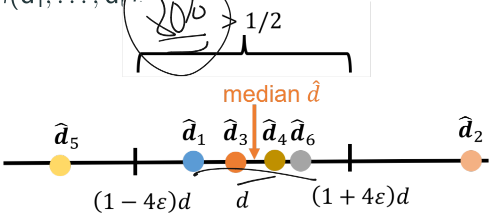

- If $> 2/3$ of trials fall in $[(1-4\epsilon)d, (1+4\epsilon)d]$, then the median will.
- Have $< 1/3$ of trials on both the left and right.

The Median Trick

- $\widehat{\mathsf{d}}_1, \ldots, \widehat{\mathsf{d}}_t$ are the outcomes of the t trials, each falling in $[(1-4\epsilon)d, (1+4\epsilon)d]$ with probability at least 4/5.
- $\widehat{\mathsf{d}} = median(\widehat{\mathsf{d}}_1, \ldots, \widehat{\mathsf{d}}_t)$.

What is the probability that the median $\widehat{\mathsf{d}}$ falls in $[(1-4\epsilon)d, (1+4\epsilon)d]$?

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$$\Pr\left(\widehat{\mathrm{d}} \notin [(1-4\epsilon)d, (1+4\epsilon)d]\right) \leq \Pr\left(X < \frac{2}{3} \cdot t\right)$$

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$$\Pr\left(\widehat{\mathrm{d}} \notin [(1-4\epsilon)d, (1+4\epsilon)d]\right) \leq \Pr\left(X < \underbrace{\frac{5}{6} \cdot \mathbb{E}[X]}\right)$$

$$< \frac{2}{3}t = \frac{5}{6} \cdot \frac{4}{5} \cdot t = \frac{5}{6}\mathbb{E}X$$

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$$\Pr\left(\widehat{\mathrm{d}} \notin [(1-4\epsilon)d, (1+4\epsilon)d]\right) \leq \Pr\left(X < \frac{5}{6} \cdot \mathbb{E}[X]\right) \leq \Pr\left(|X - \mathbb{E}[X]| \geq \frac{1}{6}\mathbb{E}[X]\right)$$

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$$\Pr\left(\widehat{\mathsf{d}} \notin [(1-4\epsilon)d, (1+4\epsilon)d]\right) \le \Pr\left(\mathsf{X} < \frac{5}{6} \cdot \mathbb{E}[\mathsf{X}]\right) \le \Pr\left(|\mathsf{X} - \mathbb{E}[\mathsf{X}]| \ge \frac{1}{6}\mathbb{E}[\mathsf{X}]\right)$$

Apply Chernoff bound:

The Median Trick

- $\widehat{\mathbf{d}}_1, \ldots, \widehat{\mathbf{d}}_\mathbf{t}$ are the outcomes of the t trials, each falling in $[(1-4\epsilon)d, (1+4\epsilon)d]$ with probability at least 4/5.
- $\widehat{\mathbf{d}} = median(\widehat{\mathbf{d}}_1, \ldots, \widehat{\mathbf{d}}_t)$.

$X_i = 1$ if $|\hat{d}_i - d| \leq 4\epsilon d$

$= 0$ otherwise.

What is the probability that the median $\widehat{\mathbf{d}}$ falls in $[(1-4\epsilon)d, (1+4\epsilon)d]$?

- Let $\mathbf{X}$ be the # of trials falling in $[(1-4\epsilon)d, (1+4\epsilon)d]$. $\mathbb{E}[\mathbf{X}] = \frac{4}{5} \cdot t$.

$$\Pr\left(\widehat{\mathbf{d}} \notin [(1-4\epsilon)d, (1+4\epsilon)d]\right) \leq \Pr\left(\mathbf{X} < \frac{5}{6} \cdot \mathbb{E}[\mathbf{X}]\right) \leq \Pr\left(|\mathbf{X} - \mathbb{E}[\mathbf{X}]| \geq \frac{1}{6}\mathbb{E}[\mathbf{X}]\right)$$

Apply Chernoff bound:

$$\Pr\left(|\mathbf{X} - \mathbb{E}[\mathbf{X}]| \geq \frac{1}{6}\mathbb{E}[\mathbf{X}]\right) \leq 2\exp\left(-\frac{\frac{1}{6}^2 \cdot \frac{4}{5}t}{2 + 1/6}\right) = \boxed{O\left(e^{-ct}\right)}.$$

The Median Trick

- $\widehat{\mathrm{d}}_1, \ldots, \widehat{\mathrm{d}}_t$ are the outcomes of the t trials, each falling in $[(1-4\epsilon)d, (1+4\epsilon)d]$ with probability at least $4/5$.
- $\widehat{\mathrm{d}} = \mathit{median}(\widehat{\mathrm{d}}_1, \ldots, \widehat{\mathrm{d}}_t)$.

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- Setting $\underline{t = O(\log(1/\delta))}$ gives failure probability $\underline{e^{-\log(1/\delta)} = \delta}$.

Median Trick

Upshot: The median of $t = O(\log(1/\delta))$ independent runs of the hashing algorithm for distinct elements returns $\widehat{\mathrm{d}} \in [(1-4\epsilon)d, (1+4\epsilon)d]$ with probability at least $1-\delta$.

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Total Space Complexity: t trials, each using $k = \frac{1}{\epsilon^2\delta'}$ hash functions, for $\delta' = 1/5$. Space is $\frac{5t}{\epsilon^2} = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$ real numbers (the minimum value of each hash function).

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A note on the median: The median is often used as a robust alternative to the mean, when there are outliers (e.g., heavy tailed distributions, corrupted data).