COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2023.

Lecture 17

Logistics

- Problem Set 3 is due Friday 11/17, 11:59pm.
- Questions about participation grade.
- Additional linear algebra review office hours – **Monday 11/13 ~~3~~:00-~~4~~:00pm**.

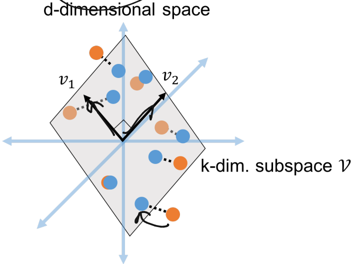
  2 3

Basic Set Up

Reminder of Set Up: Assume that $\vec{x}_1, \ldots, \vec{x}_n$ lie close to any k-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^d$. Let $\mathbf{X} \in \mathbb{R}^{n \times d}$ be the data matrix.

d: 3

k: 2



Let $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and $\mathbf{V} \in \mathbb{R}^{d \times k}$ be the matrix with these vectors as its columns.

- $\mathbf{V}\mathbf{V}^T \in \mathbb{R}^{d \times d}$ is the projection matrix onto $\mathcal{V}$.
- $\mathbf{X}\mathbf{V}\mathbf{V}^T$ gives the closest approximation to $\mathbf{X}$ with rows in $\mathcal{V}$.
- The rows of $\underbrace{\mathbf{X}\mathbf{V}\mathbf{V}^T}$ are approximations to our input points in $\mathcal{V}$. n×d
- n×k The rows of $\underbrace{\mathbf{X}\mathbf{V}}$ are compressions of these approximate points.

Last Class

argmin $x^2+2 = 0$

argmax $f(x)$

- V minimizing $\|X - XVV^T\|_F^2$ is given by:

$$\underset{\text{orthonormal } V \in \mathbb{R}^{d \times k}}{\arg\min} \|X - XVV^T\|_F^2 = \underset{\text{orthonormal } V \in \mathbb{R}^{d \times k}}{\arg\max} \|XV\|_F^2$$

- This optimal V can be found greedily. Equivalently, by computing the top k eigenvectors of X^TX.

Last Class

- $\mathbf{V}$ minimizing $\|\mathbf{X} - \mathbf{X}\mathbf{V}\mathbf{V}^T\|_F^2$ is given by:

$$\underset{\text{orthonormal } \mathbf{V} \in \mathbb{R}^{d \times k}}{\arg\min} \|\mathbf{X} - \mathbf{X}\mathbf{V}\mathbf{V}^T\|_F^2 = \underset{\text{orthonormal } \mathbf{V} \in \mathbb{R}^{d \times k}}{\arg\max} \|\mathbf{X}\mathbf{V}\|_F^2$$

- This optimal $\mathbf{V}$ can be found greedily. Equivalently, by computing the top k eigenvectors of $\mathbf{X}^T\mathbf{X}$.

↳ k eigenvectors corresponding largest eigenvalues

This Class:

- Finish up discussion of how optimal $\mathbf{V}$ is computed via eigendecomposition.
- How do we assess the error of this optimal $\mathbf{V}$.
- Connection to the singular value decomposition.

Solution via Eigendecomposition

V maximizing $\|\mathbf{XV}\|_F^2$ is given by: $\nearrow$ $\|\mathbf{X}\underline{\mathbf{V}\mathbf{V}^T}\|_F^2 = \|\mathbf{XV}\|_F^2$ $\quad \mathbf{X}\mathbf{V}\cancel{\mathbf{V}^T\mathbf{V}}^{I}$ $\quad \mathbf{XV}$

$$\underset{\text{orthonormal } \mathbf{V}\in\mathbb{R}^{d\times k}}{\arg\max} \|\mathbf{XV}\|_F^2 = \sum_{j=1}^{k} \underline{\|\mathbf{X}\vec{v}_j\|_2^2} = \sum_{j=1}^{k} \underline{\vec{v}_j^T\mathbf{X}^T\mathbf{X}\vec{v}_j}$$

Can find the columns of **V**, $\vec{v}_1, \ldots, \vec{v}_k$ greedily.

$v_1^T X^T X v_1 = \|Xv_1\|_2^2 = \lambda_1$

$$\vec{v}_1 = \underset{\vec{v} \text{ with } \|v\|_2=1}{\arg\max} \|\mathbf{X}\vec{v}\|_2^2. \quad v^T X^T X v$$

$\|Xv_i\|_2^2 = \lambda_i$

$$\vec{v}_2 = \underset{\vec{v} \text{ with } \|v\|_2=1,\ \langle\vec{v},\vec{v}_1\rangle=0}{\arg\max} \vec{v}^T\mathbf{X}^T\mathbf{X}\vec{v}.$$

$$\cdots$$

$$\vec{v}_k = \underset{\vec{v} \text{ with } \|v\|_2=1,\ \langle\vec{v},\vec{v}_j\rangle=0\ \forall j<k}{\arg\max} \vec{v}^T\mathbf{X}^T\mathbf{X}\vec{v}.$$

$\vec{v}_1, \ldots, \vec{v}_k$ are the top k eigenvectors of $\mathbf{X}^T\mathbf{X}$ by the *Courant-Fischer Principle.*

$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n\times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: orthogonal basis for subspace $\mathcal{V}$. $\mathbf{V} \in \mathbb{R}^{d\times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

Solution via Eigendecomposition

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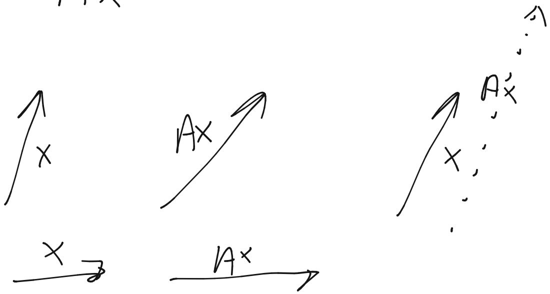
Review of Eigenvectors and Eigendecomposition

Eigenvector: $\vec{x} \in \mathbb{R}^d$ is an eigenvector of a matrix $\mathbf{A} \in \mathbb{R}^{d \times d}$ if $\mathbf{A}\vec{x} = \lambda\vec{x}$ for some scalar λ (the eigenvalue corresponding to $\vec{x}$).

Ax



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- That is, $\mathbf{A}$ just 'stretches' x.

symmetric: $A_{ij} = A_{ji} \; \forall i,j$ $\quad (X^T X)^T = X^T (X^T)^T = X^T X$

$A^T = A$ **Eigenvector:** $\vec{x} \in \mathbb{R}^d$ is an eigenvector of a matrix $\mathbf{A} \in \mathbb{R}^{d \times d}$ if $\mathbf{A}\vec{x} = \lambda \vec{x}$ for some scalar λ (the eigenvalue corresponding to $\vec{x}$).

- That is, $\mathbf{A}$ just 'stretches' x. $\quad Ix = x \cdot \lambda \quad \lambda = 1$
- If $\mathbf{A}$ is symmetric, can find d orthonormal eigenvectors $\vec{v}_1, \ldots, \vec{v}_d$. Let $\mathbf{V} \in \mathbb{R}^{d \times d}$ have these vectors as columns. $e_1, \ldots, e_d$
- $\lambda_1 \ldots \lambda_d$ are always real.

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$$\mathbf{A}\mathbf{V} = \begin{bmatrix} | & | & | & | \\ \mathbf{A}\vec{v}_1 & \mathbf{A}\vec{v}_2 & \cdots & \mathbf{A}\vec{v}_d \\ | & | & | & | \end{bmatrix}$$

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- If $\mathbf{A}$ is symmetric, can find d orthonormal eigenvectors $\vec{v}_1, \ldots, \vec{v}_d$. Let $\mathbf{V} \in \mathbb{R}^{d \times d}$ have these vectors as columns.

$$\mathbf{AV} = \begin{bmatrix} | & | & | & | \\ \mathbf{A}\vec{v}_1 & \mathbf{A}\vec{v}_2 & \cdots & \mathbf{A}\vec{v}_d \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda\vec{v}_d \\ | & | & | & | \end{bmatrix} = \mathbf{V}\boldsymbol{\Lambda}$$

$$\begin{bmatrix} v_1 & \cdots & v_d \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_d \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \cdots & \lambda_d v_d \end{bmatrix}$$

Review of Eigenvectors and Eigendecomposition

Eigenvector: $\vec{x} \in \mathbb{R}^d$ is an eigenvector of a matrix $\mathbf{A} \in \mathbb{R}^{d\times d}$ if $\mathbf{A}\vec{x} = \lambda\vec{x}$ for some scalar λ (the eigenvalue corresponding to $\vec{x}$).

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Yields eigendecomposition: $\mathbf{A}\mathbf{V}\mathbf{V}^T = \mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$.

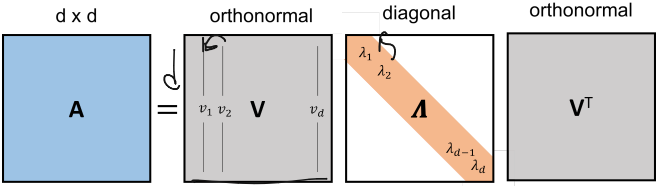
Review of Eigenvectors and Eigendecomposition

$V \quad V^T = V^{-1}$

$V^T V = I$

$VV = I$ if $k = d$

d



Typically order the eigenvectors in decreasing order:
$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$.

$V_1 \quad V_2 \quad \ldots \quad V_d$

I

$A VV^T = A$

projection onto column span V

$\Downarrow$

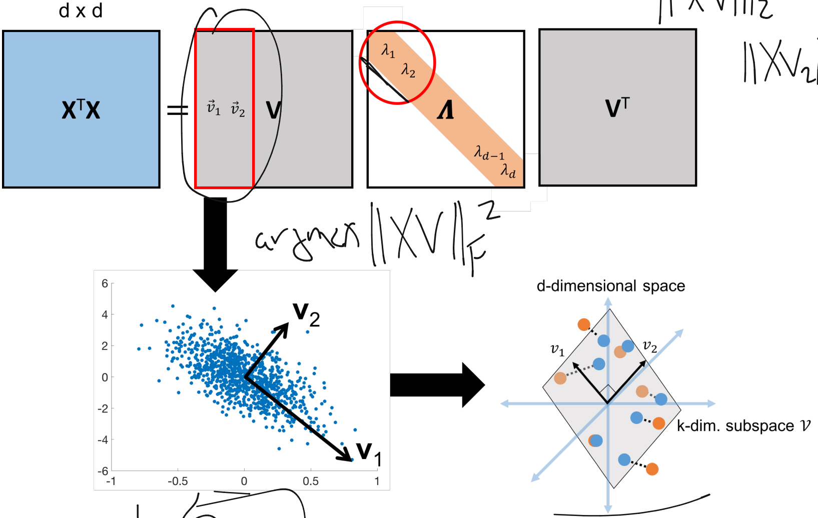
$\mathbb{R}^d$

-no-op

Low-Rank Approximation via Eigendecomposition

$\|Xv_1\|_2^2$

$\|Xv_2\|_2^2$



$\arg\max \|XV\|_F^2$

PCA

Low-Rank Approximation via Eigendecomposition

Upshot: Letting $\mathbf{V}_k$ have columns $\vec{v}_1, \ldots, \vec{v}_k$ corresponding to the top k eigenvectors of the covariance matrix $\mathbf{X}^T\mathbf{X}$, $\mathbf{V}_k$ is the orthogonal basis minimizing

$$\|\mathbf{X} - \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T\|_F^2.$$

$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: top eigenvectors of $\mathbf{X}^T\mathbf{X}$, $\mathbf{V}_k \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

Low-Rank Approximation via Eigendecomposition

Upshot: Letting $\mathbf{V}_k$ have columns $\vec{v}_1, \ldots, \vec{v}_k$ corresponding to the top k eigenvectors of the covariance matrix $\mathbf{X}^T\mathbf{X}$, $\mathbf{V}_k$ is the orthogonal basis minimizing

$$\|\mathbf{X} - \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T\|_F^2.$$

This is principal component analysis (PCA).

$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: top eigenvectors of $\mathbf{X}^T\mathbf{X}$, $\mathbf{V}_k \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

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How accurate is this low-rank approximation?

$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n\times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: top eigenvectors of $\mathbf{X}^T\mathbf{X}$, $\mathbf{V}_k \in \mathbb{R}^{d\times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

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This is principal component analysis (PCA).

How accurate is this low-rank approximation? Can understand using eigenvalues of $\mathbf{X}^T\mathbf{X}$.

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Spectrum Analysis

Let $\vec{v}_1, \ldots, \vec{v}_k$ be the top k eigenvectors of $\mathbf{X}^T\mathbf{X}$ (the top k principal components). Approximation error is:

$$\|\mathbf{X} - \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T\|_F^2$$

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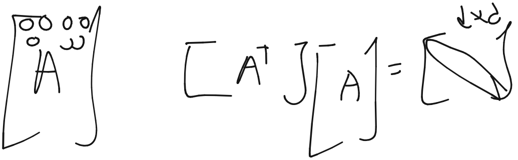

- **Problem Set:** For any matrix $\mathbf{A}$, $\|\mathbf{A}\|_F^2 = \sum_{i=1}^{d} \|\vec{a}_i\|_2^2 = \text{tr}(\mathbf{A}^T\mathbf{A})$ (sum of diagonal entries = sum eigenvalues).

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$$\|\mathbf{X} - \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T\|_F^2 = \underline{\mathrm{tr}(\mathbf{X}^T\mathbf{X})} - \mathrm{tr}(\mathbf{V}_k^T\mathbf{X}^T\mathbf{X}\mathbf{V}_k)$$

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Spectrum Analysis

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$$\|\mathbf{X} - \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T\|_F^2 = \mathbf{tr}(\mathbf{X}^T\mathbf{X}) - \mathbf{tr}(\mathbf{V}_k^T\mathbf{X}^T\mathbf{X}\mathbf{V}_k)$$

$$= \sum_{i=1}^{d} \lambda_i(\mathbf{X}^T\mathbf{X}) - \sum_{i=1}^{k} \vec{v}_i^T\mathbf{X}^T\mathbf{X}\vec{v}_i$$

$\begin{bmatrix} v_1^T \\ V_k^T \end{bmatrix} \left[X^T X \right] \left[v_1 \; V_k \right]$

$v_1^T X^T X v_1$

$[X^TX][v_i] = \lambda \cdot v_i$

$v_i^T X^T X v_i$

$= \lambda_i(X^TX) \cdot v_i^T v_i = \lambda_i(X^TX)$

$v_i^T \lambda_i(X^TX) \cdot v_i$ because v_i is eigvector of X^TX

$\|Xv_i\|_2^2$

$\| \; \lambda_i(X^TX)$

- **Problem Set:** For any matrix $\mathbf{A}$, $\|\mathbf{A}\|_F^2 = \sum_{i=1}^{d} \|\vec{a}_i\|_2^2 = \mathbf{tr}(\mathbf{A}^T\mathbf{A})$ (sum of diagonal entries = sum eigenvalues).

> $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: top eigenvectors of $\mathbf{X}^T\mathbf{X}$, $\mathbf{V}_k \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

$X^TXv = \lambda v \qquad X^TX(-v) = -\lambda v = \lambda(-v)$

Spectrum Analysis

Let $\vec{v}_1, \ldots, \vec{v}_k$ be the top k eigenvectors of $\mathbf{X}^T\mathbf{X}$ (the top k principal components). Approximation error is:

$$\begin{aligned}
\|\mathbf{X} - \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T\|_F^2 &= \operatorname{tr}(\mathbf{X}^T\mathbf{X}) - \operatorname{tr}(\mathbf{V}_k^T\mathbf{X}^T\mathbf{X}\mathbf{V}_k) \\
&= \sum_{i=1}^{d} \lambda_i(\mathbf{X}^T\mathbf{X}) - \sum_{i=1}^{k} \vec{v}_i^T\mathbf{X}^T\mathbf{X}\vec{v}_i \\
&= \sum_{i=1}^{d} \lambda_i(\mathbf{X}^T\mathbf{X}) - \sum_{i=1}^{k} \lambda_i(\mathbf{X}^T\mathbf{X})
\end{aligned}$$

- **Problem Set:** For any matrix $\mathbf{A}$, $\|\mathbf{A}\|_F^2 = \sum_{i=1}^{d} \|\vec{a}_i\|_2^2 = \operatorname{tr}(\mathbf{A}^T\mathbf{A})$ (sum of diagonal entries = sum eigenvalues).

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$$\boxed{\|\mathbf{X} - \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T\|_F^2} = \text{tr}(\mathbf{X}^T\mathbf{X}) - \text{tr}(\mathbf{V}_k^T\mathbf{X}^T\mathbf{X}\mathbf{V}_k)$$

$$= \sum_{i=1}^{d} \lambda_i(\mathbf{X}^T\mathbf{X}) - \sum_{i=1}^{k} \vec{v}_i^T\mathbf{X}^T\mathbf{X}\vec{v}_i$$

$$= \sum_{i=1}^{d} \lambda_i(\mathbf{X}^T\mathbf{X}) - \sum_{i=1}^{k} \lambda_i(\mathbf{X}^T\mathbf{X}) = \boxed{\sum_{i=k+1}^{d} \lambda_i(\mathbf{X}^T\mathbf{X})}$$

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Spectrum Analysis

Claim: The error in approximating $\mathbf{X}$ with the best rank k approximation (projecting onto the top k eigenvectors of $\mathbf{X}^T\mathbf{X}$) is:

$$\|\mathbf{X} - \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T\|_F^2 = \sum_{i=k+1}^{d} \lambda_i(\mathbf{X}^T\mathbf{X})$$

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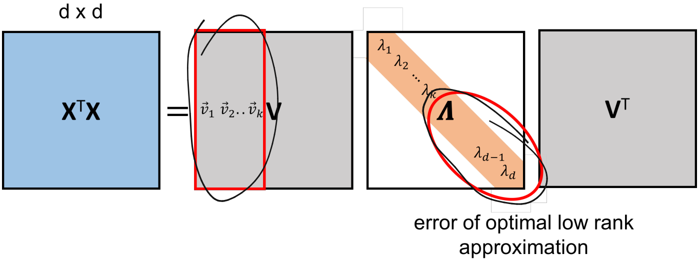
Claim: The error in approximating $\mathbf{X}$ with the best rank k approximation (projecting onto the top k eigenvectors of $\mathbf{X}^T\mathbf{X}$) is:

\# non-zero eigenvalues = rank(X^TX) = rank(X)

$$\|\mathbf{X} - \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T\|_F^2 = \sum_{i=k+1}^{d} \lambda_i(\mathbf{X}^T\mathbf{X})$$



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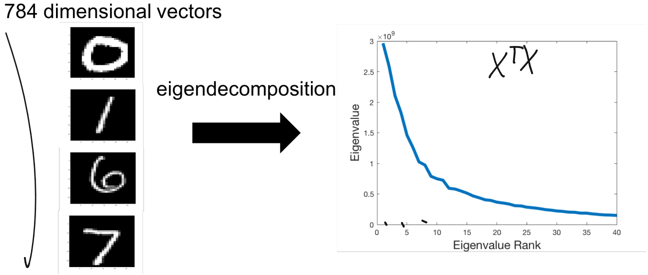
Claim: The error in approximating $\mathbf{X}$ with the best rank k approximation (projecting onto the top k eigenvectors of $\mathbf{X}^T\mathbf{X}$) is:

784

10000 X

$$\|\mathbf{X} - \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T\|_F^2 = \sum_{i=k+1}^{d} \lambda_i(\mathbf{X}^T\mathbf{X})$$



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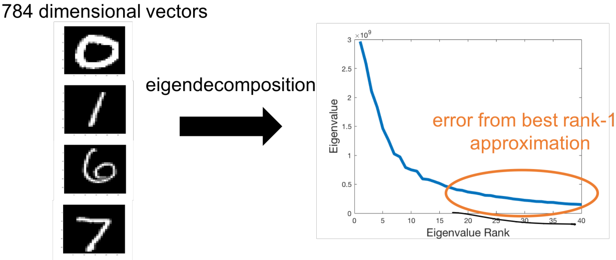
$$\|\mathbf{X} - \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T\|_F^2 = \sum_{i=k+1}^{d} \lambda_i(\mathbf{X}^T\mathbf{X})$$



> $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: top eigenvectors of $\mathbf{X}^T\mathbf{X}$, $\mathbf{V}_k \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

Spectrum Analysis

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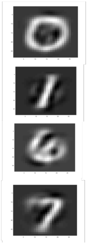
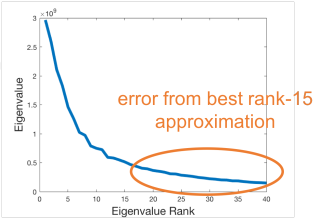
$$\|\mathbf{X} - \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T\|_F^2 = \sum_{i=k+1}^{d} \lambda_i(\mathbf{X}^T\mathbf{X})$$

784 dimensional vectors



eigendecomposition



$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: top eigenvectors of $\mathbf{X}^T\mathbf{X}$, $\mathbf{V}_k \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

Spectrum Analysis

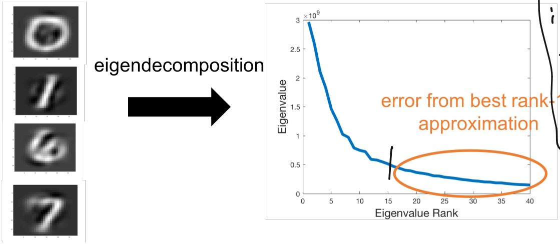
Claim: The error in approximating $\mathbf{X}$ with the best rank k approximation (projecting onto the top k eigenvectors of $\mathbf{X}^T\mathbf{X}$) is:

$$\|\mathbf{X} - \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T\|_F^2 = \underbrace{\sum_{i=k+1}^{d} \lambda_i(\mathbf{X}^T\mathbf{X})}_{} \geq 0$$

Exercise

$$\frac{\|X - XV_kV_k^T\|_F^2}{\|X\|_F^2} = \frac{\sum_{i=k+1}^{d} \lambda_i(X^TX)}{\sum_{i=1}^{d} \lambda_i(X^TX)}$$

784 dimensional vectors



- Choose k to balance accuracy/compression – often at an 'elbow'.

> $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: top eigenvectors of $\mathbf{X}^T\mathbf{X}$, $\mathbf{V}_k \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

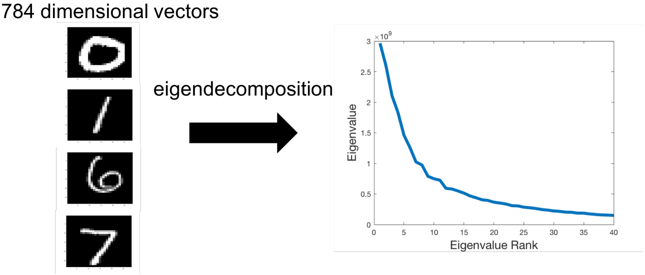
Spectrum Analysis

Plotting the spectrum of $\mathbf{X}^T\mathbf{X}$ (its eigenvalues) shows how compressible $\mathbf{X}$ is using low-rank approximation (i.e., how close $\vec{x}_1, \ldots, \vec{x}_n$ are to a low-dimensional subspace).

$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: top eigenvectors of $\mathbf{X}^T\mathbf{X}$, $\mathbf{V}_k \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

Spectrum Analysis

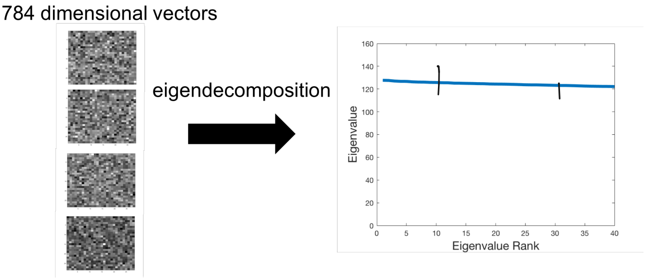
Plotting the spectrum of X^TX (its eigenvalues) shows how compressible X is using low-rank approximation (i.e., how close $\vec{x}_1, \ldots, \vec{x}_n$ are to a low-dimensional subspace).



$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $X \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: top eigenvectors of X^TX, $V_k \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

Spectrum Analysis

Plotting the spectrum of $\mathbf{X}^T\mathbf{X}$ (its eigenvalues) shows how compressible $\mathbf{X}$ is using low-rank approximation (i.e., how close $\vec{x}_1, \ldots, \vec{x}_n$ are to a low-dimensional subspace).



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d^3 $O(dk)$

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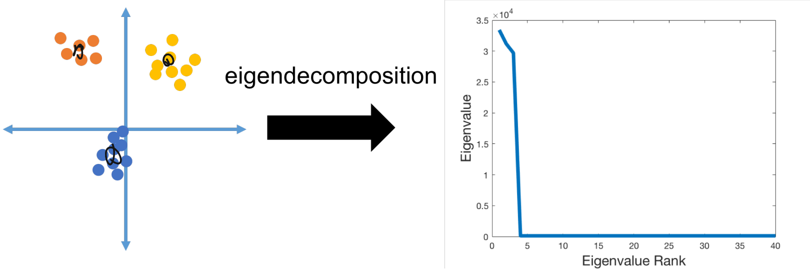

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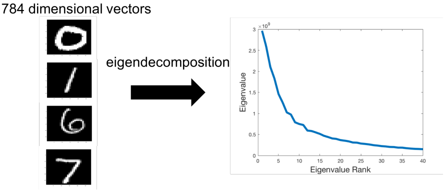
Spectrum Analysis



Exercises:

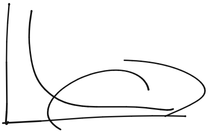
1. Show that the eigenvalues of $\mathbf{X}^T\mathbf{X}$ are always ~~positive~~ non-negative. **Hint:** Use that $\lambda_j = \vec{v}_j^T \mathbf{X}^T \mathbf{X} \vec{v}_j$.
2. Show that for symmetric $\mathbf{A}$, the trace is the sum of eigenvalues: $\text{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i(\mathbf{A})$. **Hint:** First prove the cyclic property of trace, that for any $\mathbf{MN}$, $\text{tr}(\mathbf{MN}) = \text{tr}(\mathbf{NM})$ and then apply this to $\mathbf{A}$'s eigendecomposition

Summary

- Many (most) datasets can be approximated via projection onto a low-dimensional subspace.
- Find this subspace via a maximization problem:

$$\max_{\text{orthonormal } \mathbf{V}} \|\mathbf{X}\mathbf{V}\|_F^2.$$

- Greedy solution via eigendecomposition of $\mathbf{X}^T\mathbf{X}$.
- Columns of $\mathbf{V}$ are the top eigenvectors of $\mathbf{X}^T\mathbf{X}$.
- Error of best low-rank approximation (compressibility of data) is determined by the tail of $\mathbf{X}^T\mathbf{X}$'s eigenvalue spectrum.



Interpretation in Terms of Correlation

Recall: Low-rank approximation is possible when our data features are correlated.

10000* bathrooms+ 10* (sq. ft.) ≈ list price

	bedrooms	bathrooms	sq.ft.	floors	list price	sale price
home 1	2	2	1800	2	200,000	195,000
home 2	4	2.5	2700	1	300,000	310,000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
home n	5	3.5	3600	3	450,000	450,000

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$[V][\Lambda][V^T]$

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$V^T V_k$

$$\begin{bmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_d^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & & \\ \vdots & & \\ 0 & & \end{bmatrix} = I_k$$

Our compressed dataset is $\mathbf{C} = \mathbf{X}\mathbf{V}_k$ where the columns of $\mathbf{V}_k$ are the top k eigenvectors of $\mathbf{X}^T\mathbf{X}$.

Observe that $\mathbf{C}^T\mathbf{C} = V_k^T X^T X V_k = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{bmatrix}$

$$V_k^T V \Lambda V^T V_k = I_k^T \Lambda I_k = [I \,;\, 0] \begin{bmatrix} \ddots & \\ & \Lambda \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$\Lambda_k = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}$$

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Our compressed dataset is $\mathbf{C} = \mathbf{X}\mathbf{V}_k$ where the columns of $\mathbf{V}_k$ are the top k eigenvectors of $\mathbf{X}^T\mathbf{X}$.

Observe that $\mathbf{C}^T\mathbf{C} = \mathbf{\Lambda}_k$

$C^T C$ is diagonal. I.e., all columns are orthogonal to each other, and correlations have been removed. Maximal compression.

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Algorithmic Considerations

Runtime to compute an optimal low-rank approximation:

$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: top eigenvectors of $\mathbf{X}^T\mathbf{X}$, $\mathbf{V}_k \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

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- Computing $\mathbf{X}^T\mathbf{X}$ requires $O(nd^2)$ time.

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Many faster iterative and randomized methods. Runtime is roughly $\tilde{O}(ndk)$ to output just to top k eigenvectors $\vec{v}_1, \ldots, \vec{v}_k$. eigs eig

- Will see in a few classes (power method, Krylov methods).
- One of the most intensively studied problems in numerical computation.

> $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: top eigenvectors of $\mathbf{X}^T\mathbf{X}$, $\mathbf{V}_k \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.