COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2023.

Lecture 15

Logistics

- Midterm grades and solutions are posted on Moodle.
- We'll hand out the midterms at the end of class.
- The class average was $\approx 30/39 = 77\%$.
- See Piazza post for more details. If you aren't happy with your grade, I'm happy to chat about strategies moving forward.

Quiz Question

Question 5

Not complete

Points out of 1.00

Flag question

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Suppose x=(1,2,3,4) and let y=(y_1,y_2,y_3,y_4) be a random vector where each y_i is independent and is distributed according to a Normal distribution with mean 0 and variance 1. What is the expected value of $\langle x, y\rangle^2$?

Answer:

Check

$\|x\|_2^2 = 30$

$1+2^2+3^2+4^2 = \underline{30}$

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad y = \begin{bmatrix} N(0,1) \\ N(0,1) \\ \vdots \\ N(0,1) \end{bmatrix} \qquad \mathbb{E}[\langle x,y\rangle^2]$$

$$Var(\langle x,y\rangle) = \mathbb{E}[\langle x,y\rangle^2] - \mathbb{E}[\langle x,y\rangle]^2$$

$$\mathbb{E}\, y_1 + \mathbb{E}\, 2y_2 + \mathbb{E}\, 3y_3 + \mathbb{E}\, 4y_4$$

$$Var(\langle x,y\rangle) = Var(y_1) + Var(2y_2) + Var(3y_3) + Var(4y_4)$$

0

Summary

Last Few Classes: The Johnson-Lindenstrauss Lemma

- Reduce n data points in any dimension d to $O\left(\frac{\log n/\delta}{\epsilon^2}\right)$ dimensions and preserve (with probability $\geq 1-\delta$) all pairwise distances up to $1 \pm \epsilon$.
- Compression is linear via multiplication with a random, data oblivious, matrix (linear compression)

$\Pi \, x = \hat{x}$

- Proved via the distributional JL-Lemma which shows that if $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ is a random matrix, $\mathbf{\Pi}\vec{y}_2 \approx \|\vec{y}\|$ for any y with high probability.
- Proof of distributional JL via linearity of expectation, linearity of variance, stability of the Gaussian distribution, and an exponential concentration bound for Chi-Squared random variables.

Summary

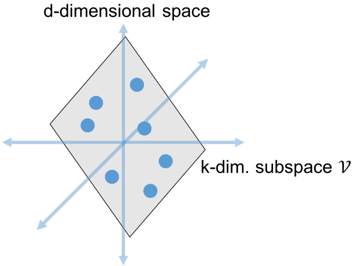
Next Few Classes: Low-rank approximation, the SVD, and principal component analysis (PCA).

- Reduce d-dimensional data points to a smaller dimension m.
- Like JL, compression is linear – by applying a matrix.
- Chose this matrix carefully, taking into account structure of the dataset.
- Can give better compression than random projection (although not directly comparable).

Will be using a fair amount of linear algebra: orthogonal basis, column/row span, eigenvectors, etc.

Embedding with Assumptions

Assume that data points $\vec{x}_1, \ldots, \vec{x}_n$ lie in any k-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^d$.

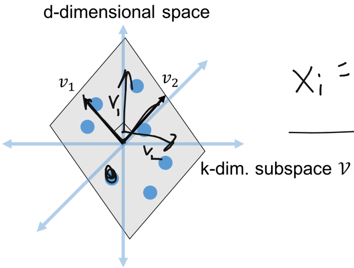


$$x_i = \left.\begin{bmatrix} \\ \\ \\ \end{bmatrix}\right\} d$$

Embedding with Assumptions

Assume that data points $\vec{x}_1, \ldots, \vec{x}_n$ $\in \mathbb{R}^d$ lie in any k-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^d$. k

$$d\left\{\overbrace{\begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix}}^{k}\right.$$



$$x_i = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

Claim: Let $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and $\mathbf{V} \in \mathbb{R}^{d \times k}$ be the matrix with these vectors as its columns. For all $\vec{x}_i, \vec{x}_j$:

$$\|\mathbf{V}^T \vec{x}_i - \mathbf{V}^T \vec{x}_j\|_2 = \|\vec{x}_i - \vec{x}_j\|_2.$$

$$k\overbrace{\begin{bmatrix} V^T \end{bmatrix}}^{d} \begin{bmatrix} x_i \end{bmatrix}_d = \begin{bmatrix} V^T x_i \end{bmatrix}_k$$

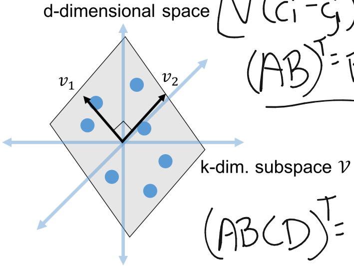
Embedding with Assumptions

Assume that data points $\vec{x}_1, \ldots, \vec{x}_n$ lie in any k-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^d$.

$$[V(c_i - c_j)]^T = (c_i - c_j)^T V^T \quad d \times 1$$

$$(AB)^T = B^T A^T$$

$$\underbrace{[V^T]}_{} [x_1] = \underset{k\times 1}{[V^T x_1]} \qquad [V^T][x_j] = \underset{k\times 1}{[V^T x_j]}$$



$$[x_i] \quad [x_j]$$

$$(ABCD)^T = D^T C^T B^T A^T$$

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$$\|\mathbf{V}^T \vec{x}_i - \mathbf{V}^T \vec{x}_j\|_2 = \|\vec{x}_i - \vec{x}_j\|_2.$$

- $\mathbf{V}^T \in \mathbb{R}^{k \times d}$ is a linear embedding of $\vec{x}_1, \ldots, \vec{x}_n$ into k dimensions with no distortion.

$$\langle z, y \rangle = z^T y = \underset{1\times d}{[\cdot\; z^T\;]} \underset{d\times 1}{[y]} = \underset{1\times 1}{[\;]}$$

$$\sum_{k=1}^{d} z(k)\, y(k) \qquad [V(c_i - c_j)]^T \qquad V(c_i - c_j)$$

Dot Product Transformation

$y = x_i - x_j \quad \|V^T y\| \leq \|V^T\| \|y\| = \|y\|$

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$$\|\mathbf{V}^T\vec{x}_i - \mathbf{V}^T\vec{x}_j\|_2^2 = \|\vec{x}_i - \vec{x}_j\|_2^2.$$

$\langle y, y \rangle = y^T y$

$V' = 2V$

$\|V'^T x_i - V'^T x_j\|_2 = 2\|x_i - x_j\|$

$\exists\, c_i, c_j \in \mathbb{R}^k$ s.t. $x_i = \underset{d\times k}{V} \underset{k\times 1}{c_i} = V_1 c_i(1) + V_2 c_i(2) \ldots$ ($d \times 1$)

$x_j = V c_j$

$= \|\underset{k\times d\ d\times k}{V^T V} c_i - V^T V c_j\|_2 = \|c_i - c_j\|_2^2 = \|x_i - x_j\|_2^2$

$V^T V = I$

$= \|Vc_i - Vc_j\|_2^2$

$\|V(c_i - c_j)\|_2^2$

$= \langle V(c_i - c_j), V(c_i - c_j) \rangle$

$(c_i - c_j)^T V^T V (c_i - c_j)$

$(c_i - c_j)^T (c_i - c_j)$

$= \|c_i - c_j\|_2^2$

$$\begin{bmatrix} v_1^T \\ \vdots \\ v_k^T \end{bmatrix} \begin{bmatrix} v_1 \cdots v_k \end{bmatrix} = (V^T V)_{ij} = \langle v_i, v_j \rangle$$

$i \neq j$ then $= 0$

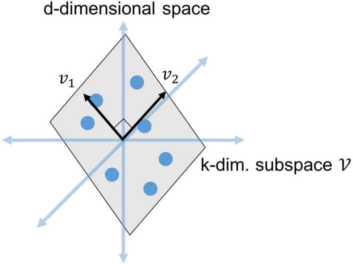
$i = j \quad \langle v_i, v_i \rangle = \|v_i\|_2^2 = 1$

$\sum_{k=1}^{d} v_i(k) \cdot v_i(k) = \sum_{k=1}^{d} v_i(k)^2 = \|v_i\|_2^2 = 1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \\ & & & 1 \end{bmatrix}$$

Embedding with Assumptions

Main Focus of Upcoming Classes: Assume that data points $\vec{x}_1, \ldots, \vec{x}_n$ lie close to any k-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^d$.



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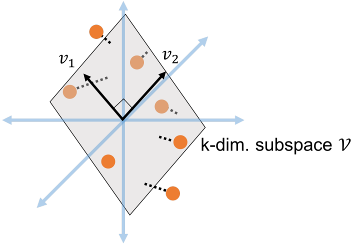

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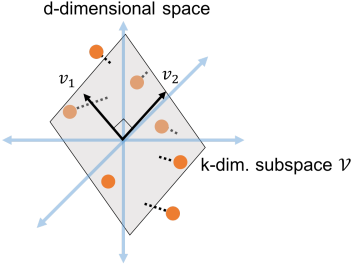

Letting $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and $\mathbf{V} \in \mathbb{R}^{d \times k}$ be the matrix with these vectors as its columns, $\mathbf{V}^T \vec{x}_i \in \mathbb{R}^k$ is still a good embedding for $x_i \in \mathbb{R}^d$.

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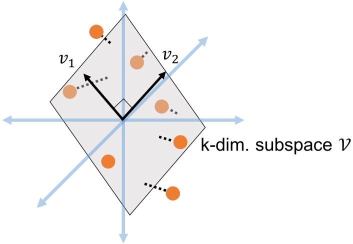

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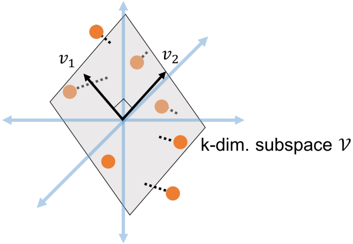

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- How do we find $\mathcal{V}$ and $\mathbf{V}$?
- How good is the embedding?

Low-Rank Factorization

Claim: $\vec{x}_1, \ldots, \vec{x}_n$ $\in \mathbb{R}^d$ lie in a k-dimensional subspace $\mathcal{V} \Leftrightarrow$ the data matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ has rank $\leq k$.

$$
\begin{array}{cc}
 & d \\
n & \begin{bmatrix} — x_1^T — \\ — x_2^T — \\ \vdots \\ X \\ \vdots \\ — x_n^T — \end{bmatrix}
\end{array}
- \operatorname{rank}(X) \leq k
$$

$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: orthogonal basis for subspace $\mathcal{V}$. $\mathbf{V} \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

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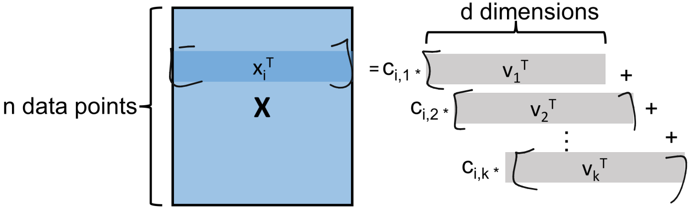
- Letting $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$, can write $\vec{x}_i$ as:

$$\vec{x}_i = \mathbf{V}\vec{c}_i = c_{i,1} \cdot \vec{v}_1 + c_{i,2} \cdot \vec{v}_2 + \ldots + c_{i,k} \cdot \vec{v}_k.$$



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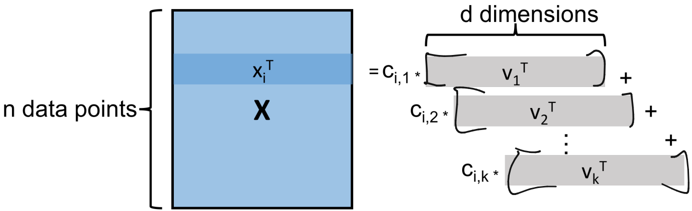
- So $\vec{v}_1, \ldots, \vec{v}_k$ span the rows of $\mathbf{X}$ and thus $\mathrm{rank}(\mathbf{X}) \leq k$.



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- Every data point $\vec{x}_i$ (row of $\mathbf{X}$) can be written as $\vec{x}_i = \mathbf{V}\vec{c}_i = c_{i,1} \cdot \vec{v}_1 + \ldots + c_{i,k} \cdot \vec{v}_k$.

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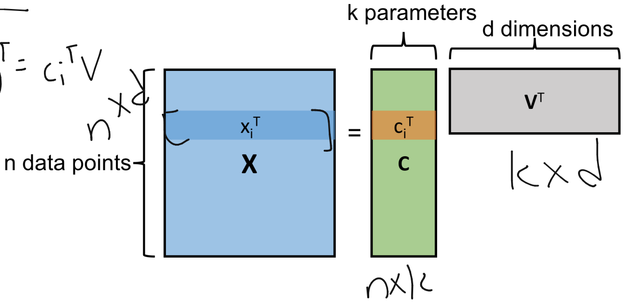
$(Vc_i)^T = c_i^T V$



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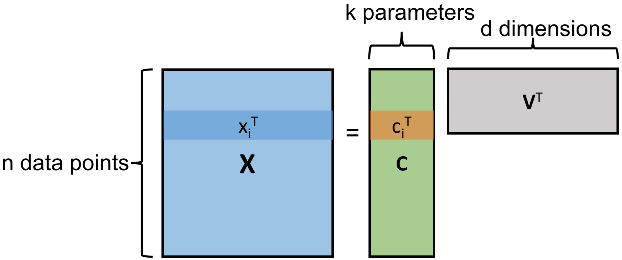

- $\mathbf{X}$ can be represented by $(n + d) \cdot k$ parameters vs. $n \cdot d$.

$\underbrace{n \cdot k}_{c_i\text{'s}}$ $\underbrace{d \cdot k}_{V}$

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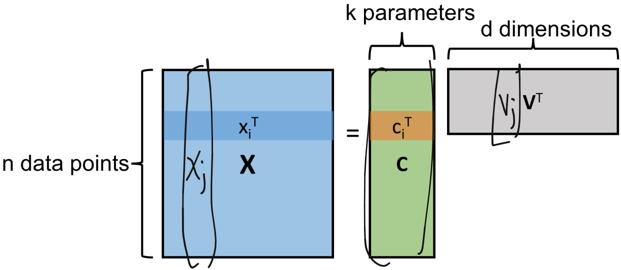

- $\mathbf{X}$ can be represented by $(n + d) \cdot k$ parameters vs. $n \cdot d$.
- The rows of $\mathbf{X}$ are spanned by k vectors: the columns of $\mathbf{V}$ $\implies$ the columns of $\mathbf{X}$ are spanned by k vectors: the columns of $\mathbf{C}$.

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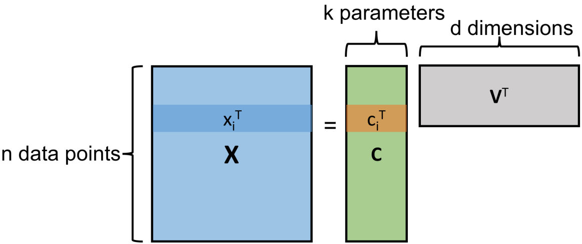
Low-Rank Factorization

Claim: If $\vec{x}_1, \ldots, \vec{x}_n$ lie in a k-dimensional subspace with orthonormal basis $\mathbf{V} \in \mathbb{R}^{d \times k}$, the data matrix can be written as $\mathbf{X} = \mathbf{C}\mathbf{V}^T$.



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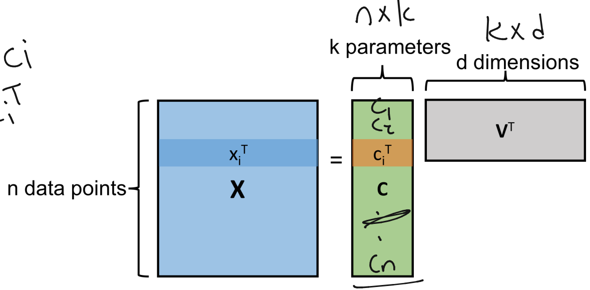
$V^T x_i = c_i$

$x_i^T V = c_i^T$

$XV = C$



Exercise: What is this coefficient matrix $\mathbf{C}$? **Hint:** Use that $\mathbf{V}^T\mathbf{V} = \mathbf{I}$.

$X = CV^T$ $\quad XV = CV^T\cancel{V}^{I}$ $\quad XV = C$

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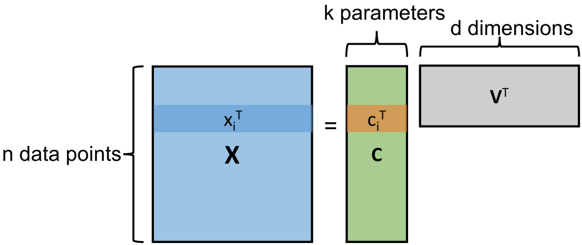

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- $\mathbf{X} = \mathbf{C}\mathbf{V}^T \implies \mathbf{X}\mathbf{V} = \mathbf{C}\mathbf{V}^T\mathbf{V}$

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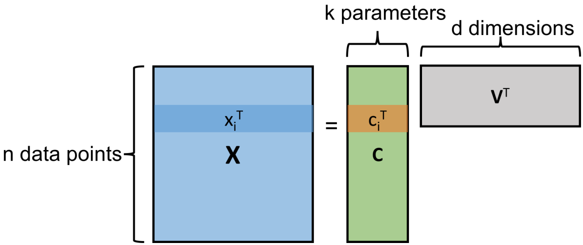

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- $\mathbf{X} = \mathbf{C}\mathbf{V}^T \implies \mathbf{X}\mathbf{V} = \mathbf{C}\overset{I}{\cancel{\mathbf{V}^T\mathbf{V}}} \implies \mathbf{X}\mathbf{V} = \mathbf{C}$

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Low-Rank Factorization

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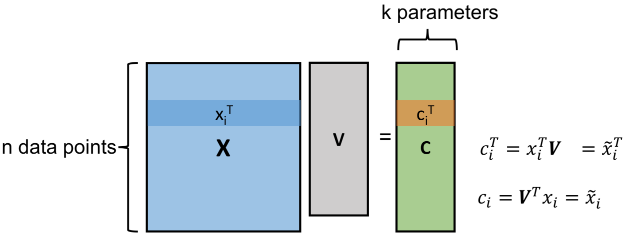

Exercise: What is this coefficient matrix $\mathbf{C}$? **Hint:** Use that $\mathbf{V}^T\mathbf{V} = \mathbf{I}$.

- $\mathbf{X} = \mathbf{C}\mathbf{V}^T \implies \mathbf{X}\mathbf{V} = \mathbf{C}\mathbf{V}^T\mathbf{V} \implies \mathbf{X}\mathbf{V} = \mathbf{C}$

$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: orthogonal basis for subspace $\mathcal{V}$. $\mathbf{V} \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

Projection View

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XV

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$$\mathbf{X} = \mathbf{X}\overbrace{\mathbf{V}\mathbf{V}^T}^{d \times d}.$$

$V^T V = I$

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- $\mathbf{V}\mathbf{V}^T$ is a projection matrix, which projects vectors onto the subspace $\mathcal{V}$.

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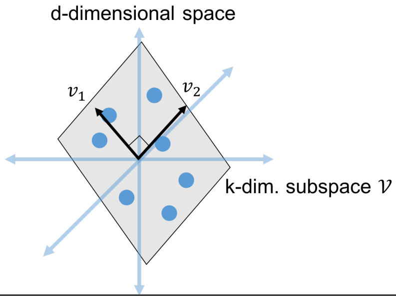

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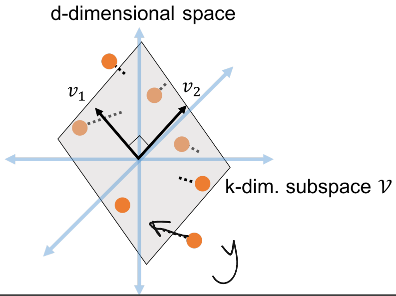
$$\overbrace{VV^T}^{d\times d}\,\overbrace{y}^{d\times 1}$$



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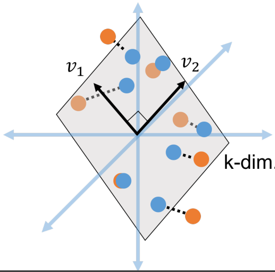

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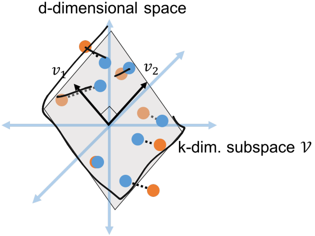
Low-Rank Approximation

Claim: If $\vec{x}_1, \ldots, \vec{x}_n$ lie close to a k-dimensional subspace $\mathcal{V}$ with orthonormal basis $\mathbf{V} \in \mathbb{R}^{d \times k}$, the data matrix can be approximated as:

$$\boxed{\mathbf{X} \approx \mathbf{X}\mathbf{V}\mathbf{V}^T}$$



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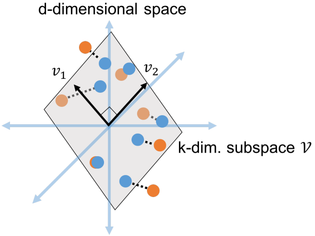

Note: $\mathbf{X}\mathbf{V}\mathbf{V}^T$ has rank k. It is a low-rank approximation of $\mathbf{X}$.

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