COMPSCI 514: Algorithms for Data Science

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Lecture 14

Logistics

- We will be grading the exams this upcoming week.
- We will release solutions shortly – we still have some students taking make up exams.
- Feel free to ask about the questions in office hours.
- Problem Set 3 will be released next week.

Summary

Last Class Prior to Exam: The Johnson-Lindenstrauss Lemma

- Intro to dimensionality reduction.
- Intro to low-distortion embeddings and the JL Lemma.
- Reduction of JL Lemma to the Distributional JL Lemma.

This Class:

- Proof the Distributional JL Lemma.
- Example application of JL to clustering.

Next Few Classes:

- Data-dependent dimensionality reduction via PCA. Formulation as low-rank matrix approximation.
- This would be a good time to review your linear algebra – matrix multiplication, dot products, subspaces, orthogonal projection, etc. See schedule tab for resources.

Distributional JL

The Johnson-Lindenstrauss Lemma is a direct consequence of a closely related lemma:

Distributional JL Lemma: Let $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$. If we set $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$, then for any $\vec{y} \in \mathbb{R}^d$, with probability $\geq 1 - \delta$

$$(1-\epsilon)\|\vec{y}\|_2 \leq \|\mathbf{\Pi}\vec{y}\|_2 \leq (1+\epsilon)\|\vec{y}\|_2.$$

Applying a random matrix $\mathbf{\Pi}$ to any vector $\vec{y}$ preserves $\vec{y}$'s norm with high probability.

- Like a low-distortion embedding, but for the length of a compressed vector rather than distances between vectors.
- Will prove today from first principles.

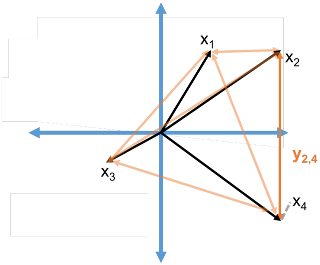
$\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection matrix. d: original dimension. m: compressed dimension, ϵ: embedding error, δ: embedding failure prob.

Distributional JL $\implies$ JL

Distributional JL Lemma $\implies$ JL Lemma: Distributional JL show that a random projection $\mathbf{\Pi}$ preserves the norm of any y. The main JL Lemma says that $\mathbf{\Pi}$ preserves distances between vectors.

Since $\mathbf{\Pi}$ is linear these are the same thing.

Proof: Given $\vec{x}_1, \ldots, \vec{x}_n$, define $\binom{n}{2}$ vectors $\vec{y}_{ij}$ where $\vec{y}_{ij} = \vec{x}_i - \vec{x}_j$. vspace-1em



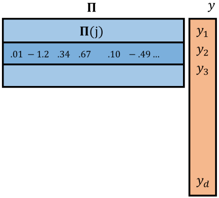
- If we choose $\mathbf{\Pi}$ with $m = O\left(\frac{\log 1/\delta'}{\epsilon^2}\right)$, for each $\vec{y}_{ij}$ with probability

Distributional JL Proof

Distributional JL Lemma: Let $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$. If we set $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$, then for any $\vec{y} \in \mathbb{R}^d$, with probability $\geq 1 - \delta$

$$(1-\epsilon)\|\vec{y}\|_2 \leq \|\mathbf{\Pi}\vec{y}\|_2 \leq (1+\epsilon)\|\vec{y}\|_2$$

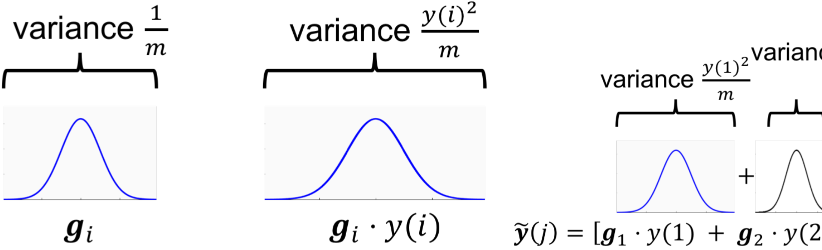
- Let $\tilde{\mathbf{y}}$ denote $\mathbf{\Pi}\vec{y}$ and let $\mathbf{\Pi}(j)$ denote the j^{th} row of $\mathbf{\Pi}$.
- For any j, $\tilde{\mathbf{y}}(j) = \langle \mathbf{\Pi}(j), \vec{y} \rangle = \sum_{i=1}^{d} \mathbf{g}_i \cdot \vec{y}(i)$ where $\mathbf{g}_i \sim \mathcal{N}(0, 1/m)$.



$\vec{y} \in \mathbb{R}^d$: arbitrary vector, $\tilde{\mathbf{y}} \in \mathbb{R}^m$: compressed vector, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection. d: original dim. m: compressed dim, ϵ: error, δ: failure prob.

Distributional JL Proof

- Let $\tilde{\mathbf{y}}$ denote $\mathbf{\Pi}\vec{y}$ and let $\mathbf{\Pi}(j)$ denote the j^{th} row of $\mathbf{\Pi}$.
- For any j, $\tilde{\mathbf{y}}(j) = \langle \mathbf{\Pi}(j), \vec{y} \rangle = \sum_{i=1}^{d} \mathbf{g}_i \cdot \vec{y}(i)$ where $\mathbf{g}_i \sim \mathcal{N}(0, 1/m)$.
- $\mathbf{g}_i \cdot \vec{y}(i) \sim \mathcal{N}(0, \frac{\vec{y}(i)^2}{m})$: normally distributed with variance $\frac{\vec{y}(i)^2}{m}$.



What is the distribution of $\tilde{\mathbf{y}}(j)$? Also Gaussian!

> $\vec{y} \in \mathbb{R}^d$: arbitrary vector, $\tilde{\mathbf{y}} \in \mathbb{R}^m$: compressed vector, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection mapping $\vec{y} \to \tilde{\mathbf{y}}$. $\mathbf{\Pi}(j)$: j^{th} row of $\mathbf{\Pi}$, d: original dimension. m: compressed dimension, $\mathbf{g}_i$: normally distributed random variable.

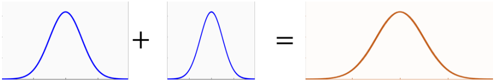
Distributional JL Proof

Letting $\tilde{\mathbf{y}} = \mathbf{\Pi}\vec{y}$, we have $\tilde{\mathbf{y}}(j) = \langle \mathbf{\Pi}(j), \vec{y} \rangle$ and:

$$\tilde{\mathbf{y}}(j) = \sum_{i=1}^{d} \mathbf{g}_i \cdot \vec{y}(i) \text{ where } \mathbf{g}_i \cdot \vec{y}(i) \sim \mathcal{N}\left(0, \frac{\vec{y}(i)^2}{m}\right).$$

Stability of Gaussian Random Variables. For independent $a \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $b \sim \mathcal{N}(\mu_2, \sigma_2^2)$ we have:

$$a + b \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$



Thus, $\tilde{\mathbf{y}}(j) \sim \mathcal{N}(0, \frac{\vec{y}(1)^2}{m} + \frac{\vec{y}(2)^2}{m} + \ldots + \frac{\vec{y}(d)^2}{m}$ $\frac{\|\vec{y}\|_2^2}{m}$) I.e., $\tilde{\mathbf{y}}$ itself is a random Gaussian vector. Rotational invariance of the Gaussian distribution.

$\vec{y} \in \mathbb{R}^d$: arbitrary vector, $\tilde{\mathbf{y}} \in \mathbb{R}^m$: compressed vector, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection mapping $\vec{y} \to \tilde{\mathbf{y}}$. $\mathbf{\Pi}(j)$: j^{th} row of $\mathbf{\Pi}$, d: original dimension. m: com-

Distributional JL Proof

So far: Letting $\mathbf{\Pi} \in \mathbb{R}^{d \times m}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$, for any $\vec{y} \in \mathbb{R}^d$, letting $\tilde{\mathbf{y}} = \mathbf{\Pi}\vec{y}$:

$$\tilde{\mathbf{y}}(j) \sim \mathcal{N}(0, \|\vec{y}\|_2^2/m).$$

What is $\mathbb{E}[\|\tilde{\mathbf{y}}\|_2^2]$?

$$\mathbb{E}[\|\tilde{\mathbf{y}}\|_2^2] = \mathbb{E}\left[\sum_{j=1}^{m} \tilde{\mathbf{y}}(j)^2\right] = \sum_{j=1}^{m} \mathbb{E}[\tilde{\mathbf{y}}(j)^2]$$

$$= \sum_{j=1}^{m} \frac{\|\vec{y}\|_2^2}{m} = \|\vec{y}\|_2^2$$

So $\tilde{\mathbf{y}}$ has the right norm in expectation.

How is $\|\tilde{\mathbf{y}}\|_2^2$ distributed? Does it concentrate?

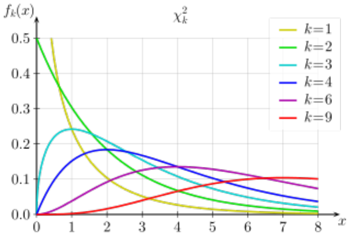
$\vec{y} \in \mathbb{R}^d$: arbitrary vector, $\tilde{\mathbf{y}} \in \mathbb{R}^m$: compressed vector, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection mapping $\vec{y} \to \tilde{\mathbf{y}}$. $\mathbf{\Pi}(j)$: j^{th} row of $\mathbf{\Pi}$, d: original dimension. m: compressed dimension, $\mathbf{g}_i$: normally distributed random variable

Distributional JL Proof

So far: Letting $\mathbf{\Pi} \in \mathbb{R}^{d \times m}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$, for any $\vec{y} \in \mathbb{R}^d$, letting $\tilde{\mathbf{y}} = \mathbf{\Pi}\vec{y}$:

$$\tilde{\mathbf{y}}(j) \sim \mathcal{N}(0, \|\vec{y}\|_2^2/m) \text{ and } \mathbb{E}[\|\tilde{\mathbf{y}}\|_2^2] = \|\vec{y}\|_2^2$$

$\|\tilde{\mathbf{y}}\|_2^2 = \sum_{i=1}^m \tilde{\mathbf{y}}(j)^2$ a Chi-Squared random variable with m degrees of freedom (a sum of m squared independent Gaussians)

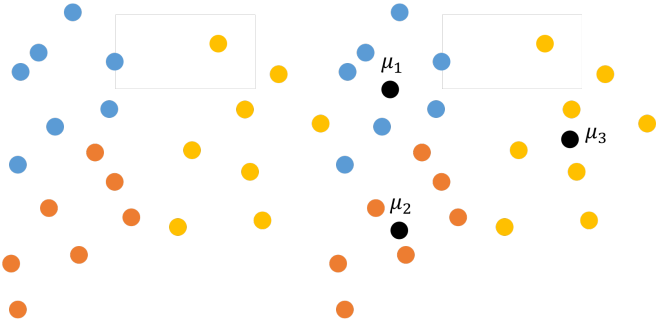


Lemma: (Chi-Squared Concentration) Letting $\mathbf{Z}$ be a Chi-Squared random variable with m degrees of freedom,

$$\Pr\left[|\mathbf{Z} - \mathbb{E}\mathbf{Z}| \geq \epsilon \mathbb{E}\mathbf{Z}\right] \leq 2e^{-m\epsilon^2/8}.$$

Example Application: k-means clustering

Goal: Separate n points in d dimensional space into k groups.



k-means Objective: $Cost(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \min_{\mathcal{C}_1, \ldots \mathcal{C}_k} \sum_{j=1}^{k} \sum_{\vec{x} \in \mathcal{C}_k} \|\vec{x} - \mu_j\|_2^2.$

Write in terms of distances:

$$Cost(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \min_{\mathcal{C}_1, \ldots \mathcal{C}_k} \sum_{j=1}^{k} \sum_{\vec{x}_1, \vec{x}_2 \in \mathcal{C}_k} \|\vec{x}_1 - \vec{x}_2\|_2^2$$

Example Application: k-means clustering

k-means Objective: $Cost(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \min_{\mathcal{C}_1, \ldots \mathcal{C}_k} \sum_{j=1}^{k} \sum_{\vec{x}_1, \vec{x}_2 \in \mathcal{C}_k} \|\vec{x}_1 - \vec{x}_2\|_2^2$

If we randomly project to $m = O\left(\frac{\log n}{\epsilon^2}\right)$ dimensions, for all pairs $\vec{x}_1, \vec{x}_2$,

$$(1-\epsilon)\|\vec{x}_1 - \vec{x}_2\|_2^2 \leq \|\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2\|_2^2 \leq (1+\epsilon)\|\vec{x}_1 - \vec{x}_2\|_2^2 \implies$$

Letting $\overline{Cost}(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \min_{\mathcal{C}_1, \ldots \mathcal{C}_k} \sum_{j=1}^{k} \sum_{\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2 \in \mathcal{C}_k} \|\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2\|_2^2$

$$(1-\epsilon)Cost(\mathcal{C}_1, \ldots, \mathcal{C}_k) \leq \overline{Cost}(\mathcal{C}_1, \ldots, \mathcal{C}_k) \leq (1+\epsilon)Cost(\mathcal{C}_1, \ldots, \mathcal{C}_k).$$

Upshot: Can cluster in m dimensional space (much more efficiently) and minimize $\overline{Cost}(\mathcal{C}_1, \ldots, \mathcal{C}_k)$. The optimal set of clusters will have true cost within $1 + c\epsilon$ times the true optimal. Good exercise to prove this.