COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2023.

Lecture 14

Logistics

- We will be grading the exams this upcoming week.
- We will release solutions shortly – we still have some students taking make up exams.
- Feel free to ask about the questions in office hours.
- Problem Set 3 will be released next week.
- Quiz due Monday.

Summary

Last Class Prior to Exam: The Johnson-Lindenstrauss Lemma

- Intro to dimensionality reduction.
- Intro to low-distortion embeddings and the JL Lemma.
- Reduction of JL Lemma to the Distributional JL Lemma.

$m = O\left(\frac{\log n}{\epsilon^2}\right)$

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- Proof the Distributional JL Lemma.
- Example application of JL to clustering.

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- Proof the Distributional JL Lemma.
- Example application of JL to clustering.

Next Few Classes:

- Data-dependent dimensionality reduction via PCA. Formulation as low-rank matrix approximation.
- This would be a good time to review your linear algebra – matrix multiplication, dot products, subspaces, orthogonal projection, etc. See schedule tab for resources.

Distributional JL

The Johnson-Lindenstrauss Lemma is a direct consequence of a closely related lemma:

Distributional JL Lemma: Let $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$. If we set $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$, then for any $\vec{y} \in \mathbb{R}^d$, with probability $\geq 1 - \delta$

$$(1 - \epsilon)\|\vec{y}\|_2 \leq \|\mathbf{\Pi}\vec{y}\|_2 \leq (1 + \epsilon)\|\vec{y}\|_2.$$

Applying a random matrix $\mathbf{\Pi}$ to any vector $\vec{y}$ preserves $\vec{y}$'s norm with high probability.

- Like a low-distortion embedding, but for the length of a compressed vector rather than distances between vectors.
- Will prove today from first principles.

$\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection matrix. d: original dimension. m: compressed dimension, ϵ: embedding error, δ: embedding failure prob.

Distributional JL Lemma $\implies$ JL Lemma: Distributional JL show that a random projection $\mathbf{\Pi}$ preserves the norm of any y. The main JL Lemma says that $\mathbf{\Pi}$ preserves distances between vectors.

Since $\mathbf{\Pi}$ is linear these are the same thing.

Proof: Given $\vec{x}_1, \ldots, \vec{x}_n$, define $\binom{n}{2}$ vectors $\vec{y}_{ij}$ where $\vec{y}_{ij} = \vec{x}_i - \vec{x}_j$.
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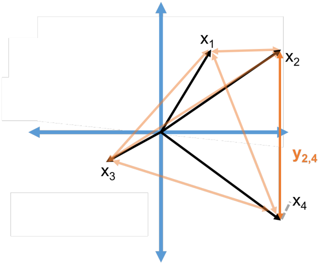
$\|x_i - x_j\|$



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- If we choose $\mathbf{\Pi}$ with $m = O\left(\frac{\log 1/\delta'}{\epsilon^2}\right)$, for each $\vec{y}_{ij}$ with probability $\geq 1 - \delta'$ we have:

$$(1-\epsilon)\|\vec{y}_{ij}\|_2 \leq \|\mathbf{\Pi}\vec{y}_{ij}\|_2 \leq (1+\epsilon)\|\vec{y}_{ij}\|_2$$

$\vec{x}_1, \ldots, \vec{x}_n$: original points, $\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_n$: compressed points, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection matrix. d: original dimension. m: compressed dimension, ϵ: embedding error, δ: embedding failure prob.

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$\Pi x_i - \Pi x_j$, $\tilde{x}_j$

$$(1-\epsilon)\|\vec{x}_i - \vec{x}_j\|_2 \leq \|\mathbf{\Pi}(\vec{x}_i - \vec{x}_j)\|_2 \leq (1+\epsilon)\|\vec{x}_i - \vec{x}_j\|_2$$

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Setting $\delta' = \delta/\binom{n}{2}$, by a union bound, this holds simultaneously for all $\vec{x}_i, \vec{x}_j$ with probability at least $1 - \delta$ for $m = O\left(\frac{\log(n/\delta)}{\epsilon^2}\right)$, giving the JL Lemma.

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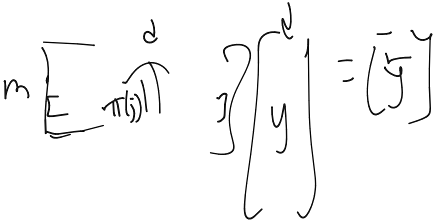

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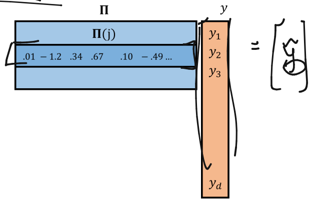

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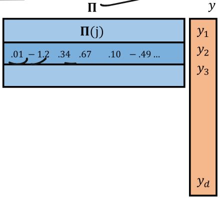

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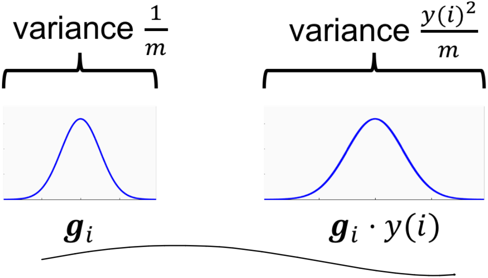

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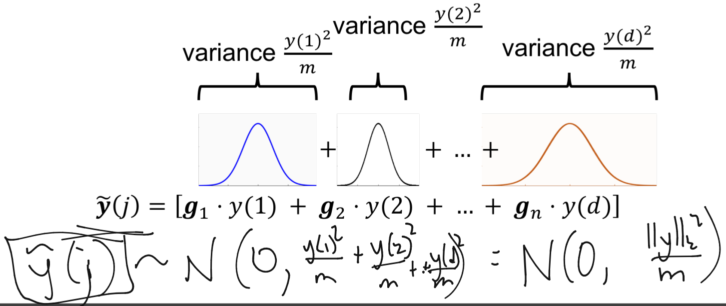

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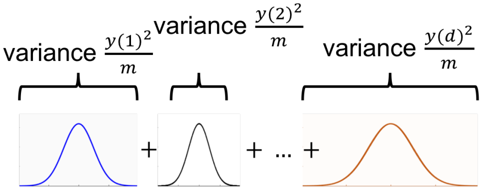

$$\tilde{\mathbf{y}}(j) = [\boldsymbol{g}_1 \cdot y(1) + \boldsymbol{g}_2 \cdot y(2) + \ldots + \boldsymbol{g}_n \cdot y(d)]$$

What is the distribution of $\tilde{\mathbf{y}}(j)$?

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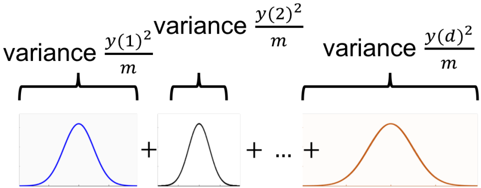

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What is the distribution of $\tilde{\mathbf{y}}(j)$? Also Gaussian!

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Stability of Gaussian Random Variables. For independent $a \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $b \sim \mathcal{N}(\mu_2, \sigma_2^2)$ we have:

$$a + b \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

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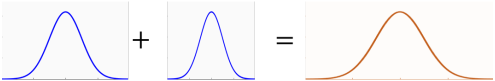

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Thus, $\tilde{\mathbf{y}}(j) \sim \mathcal{N}(0, \frac{\vec{y}(1)^2}{m} + \frac{\vec{y}(2)^2}{m} + \ldots + \frac{\vec{y}(d)^2}{m})$

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$$a + b \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Thus, $\tilde{\mathbf{y}}(j) \sim \mathcal{N}\left(0, \frac{\|\vec{y}\|_2^2}{m}\right)$

$\vec{y} \in \mathbb{R}^d$: arbitrary vector, $\tilde{\mathbf{y}} \in \mathbb{R}^m$: compressed vector, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection mapping $\vec{y} \to \tilde{\mathbf{y}}$. $\mathbf{\Pi}(j)$: j^{th} row of $\mathbf{\Pi}$, d: original dimension. m: compressed dimension, $\mathbf{g}_i$: normally distributed random variable

Distributional JL Proof

Letting $\tilde{\mathbf{y}} = \mathbf{\Pi}\vec{y}$, we have $\tilde{\mathbf{y}}(j) = \langle \mathbf{\Pi}(j), \vec{y} \rangle$ and:

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Thus, $\tilde{\mathbf{y}}(j) \sim \mathcal{N}(0, \frac{\|\vec{y}\|_2^2}{m})$ I.e., $\tilde{\mathbf{y}}$ itself is a random Gaussian vector. Rotational invariance of the Gaussian distribution.

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$g_i \sim N(0, \frac{1}{m})$

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So $\tilde{\mathbf{y}}$ has the right norm in expectation.

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How is $\|\tilde{\mathbf{y}}\|_2^2$ distributed? Does it concentrate?

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gaussian

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$\|\tilde{\mathbf{y}}\|_2^2 = \sum_{i=1}^m \tilde{\mathbf{y}}(j)^2$ a Chi-Squared random variable with m degrees of freedom (a sum of m squared independent Gaussians)

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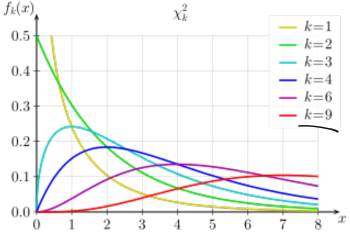

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Lemma: (Chi-Squared Concentration) Letting $\mathbf{Z}$ be a Chi-Squared random variable with m degrees of freedom,

$$\Pr\left[|\mathbf{Z} - \mathbb{E}\mathbf{Z}| \geq \epsilon \mathbb{E}\mathbf{Z}\right] \leq 2e^{-m\epsilon^2/8}.$$

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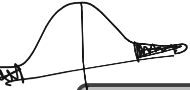

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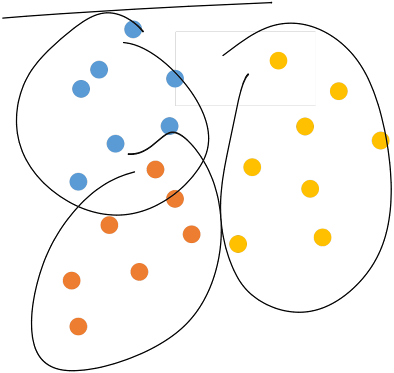
Gives the distributional JL Lemma and thus the classic JL Lemma!

Example Application: k-means clustering

Goal: Separate n points in d dimensional space into k groups.



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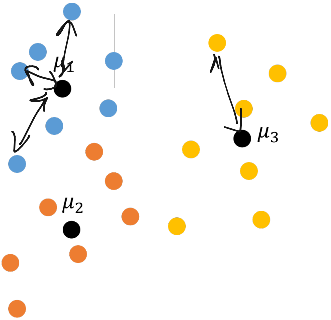

k-means Objective: $Cost(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \min_{\mathcal{C}_1, \ldots \mathcal{C}_k} \sum_{j=1}^{k} \sum_{\vec{x} \in \mathcal{C}_k} \|\vec{x} - \mu_j\|_2^2.$

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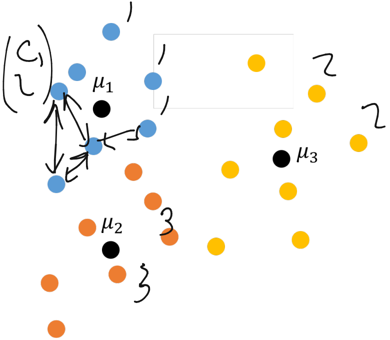

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Write in terms of distances:

$$Cost(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \min_{\mathcal{C}_1,\ldots\mathcal{C}_k} \sum_{j=1}^{k} \sum_{\vec{x}_1,\vec{x}_2 \in \mathcal{C}_k} \|\vec{x}_1 - \vec{x}_2\|_2^2$$

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If we randomly project to $m = O\left(\frac{\log n}{\epsilon^2}\right)$ dimensions, for all pairs $\vec{x}_1, \vec{x}_2$,

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Letting $\overline{Cost}(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \min_{\mathcal{C}_1, \ldots \mathcal{C}_k} \sum_{j=1}^{k} \sum_{\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2 \in \mathcal{C}_k} \|\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2\|_2^2$

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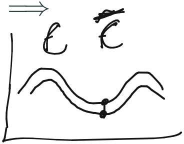

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Upshot: Can cluster in m dimensional space (much more efficiently) and minimize $\overline{Cost}(\mathcal{C}_1, \ldots, \mathcal{C}_k)$. The optimal set of clusters will have true cost within $1 + c\epsilon$ times the true optimal. Good exercise to prove this.