COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2023.

Lecture 10

Logistics

- Problem Set 2 is due Monday 10/16 at 11:59pm.
- The midterm is in class on Tuesday 10/24. Midterm study material will be posted shortly.
- We have a quiz this week, but not the next two weeks (due to the problem set and midterm).

– I away next week

Summary

Last Class:

- Discussion of practical algorithms for distinct items estimation (LogLog/HyperLogLog). 1010 000
- Introduction of Jaccard similarity and the similarity research problem.

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- Discussion of practical algorithms for distinct items estimation (LogLog/HyperLogLog).
- Introduction of Jaccard similarity and the similarity research problem.

This Class:

- Locality sensitive hashing for fast similarity search.
- MinHash as a locality sensitive hash function for Jaccard similarity
- Balancing false positives and negatives with LSH signatures and repeated hash tables.

Search with Jaccard Similarity

(A, B, C) (C, D, E) $\frac{1}{5}$

$$J(A,B) = \frac{|A \cap B|}{|A \cup B|} = \frac{\#\text{ shared elements}}{\#\text{ total elements}}.$$

Want Fast Implementations For:

- **Near Neighbor Search:** Have a database of n sets/bit strings and given a set A, want to find if it has high Jaccard similarity to anything in the database. $\Omega(n)$ time with a linear scan.
- **All-pairs Similarity Search:** Have n different sets/bit strings and want to find all pairs with high Jaccard similarity. $\Omega(n^2)$ time if we check all pairs explicitly.

Will speed up via randomized locality sensitive hashing.

approximate matching

Locality Sensitive Hashing

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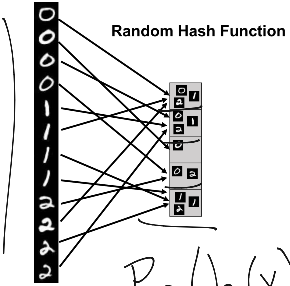
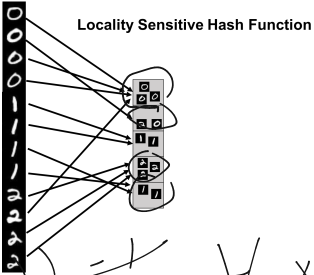
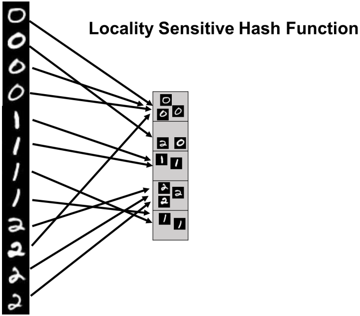




$Pr(h(x) = h(y)) = \frac{1}{n} \quad \forall\, x \neq y$

LSH For Similarity Search

How does locality sensitive hashing (LSH) help with similarity search?



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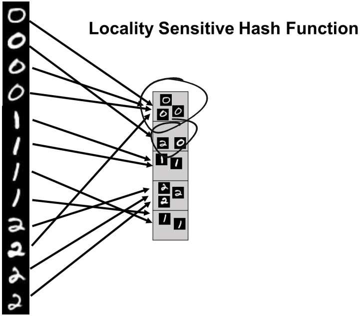

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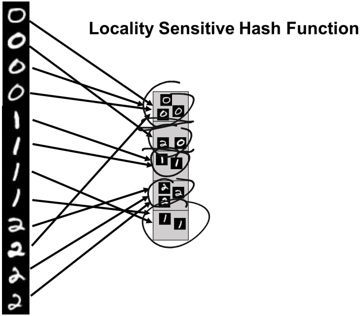

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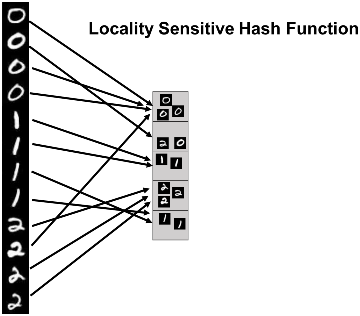

- **Near Neighbor Search:** Given item x, compute $\mathbf{h}(x)$. Only search for similar items in the $\mathbf{h}(x)$ bucket of the hash table.
- **All-pairs Similarity Search:** Scan through all buckets of the hash table and look for similar pairs within each bucket.
- We will use $\mathbf{h}(x) = \mathbf{g}(MinHash(x))$ where $\mathbf{g} : [0, 1] \to [n]$ is a random hash function. Why?

MinHashing

An Example: Locality sensitive hashing for Jaccard similarity.

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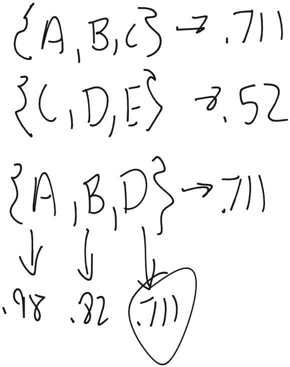
MinHash(A): [Andrei Broder, 1997 at Altavista]

- Let $\mathbf{h} : U \rightarrow [0, 1]$ be a random hash function
- $\mathbf{s} := 1$
- For $x_1, \ldots, x_{|A|} \in A$
    - $\mathbf{s} := \min(\mathbf{s}, \mathbf{h}(x_k))$
- Return $\mathbf{s}$



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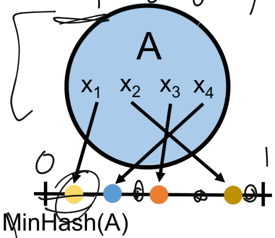

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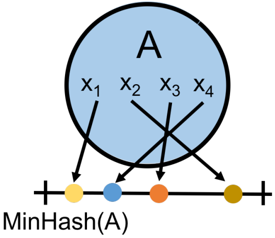

Identical to our distinct elements sketch!

MinHash Analysis

For two sets A and B, what is $\Pr(MinHash(A) = MinHash(B))$?

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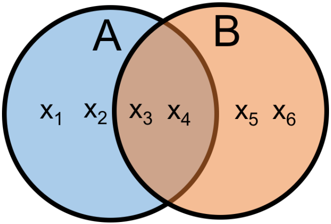
$J(A,B) = \frac{2}{6} = \frac{1}{3}$



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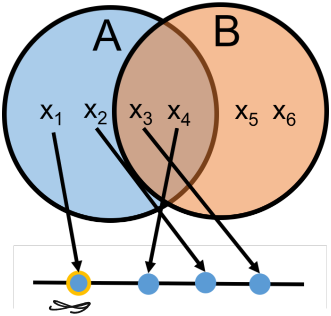

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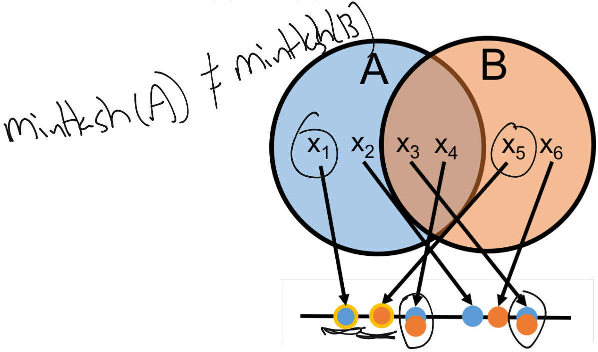

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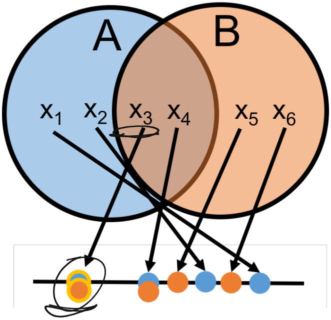

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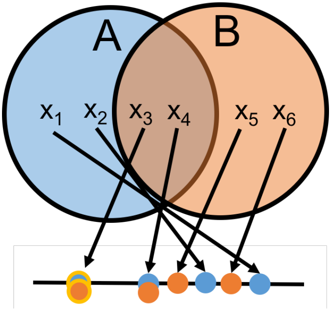

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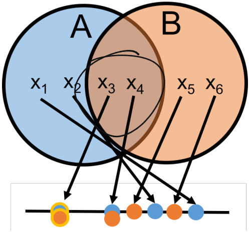
For two sets A and B, what is $\Pr(MinHash(A) = MinHash(B))$? $= J(A,B)$

Claim: $MinHash(A) = MinHash(B)$ only if an item in $A \cap B$ has the minimum hash value in both sets.



$$\frac{|A \cap B|}{|A \cup B|} = J(A,B)$$

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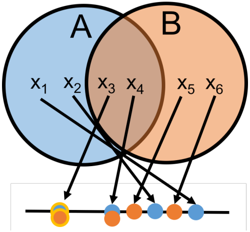

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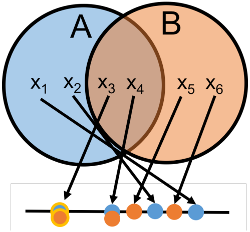

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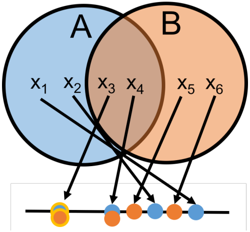

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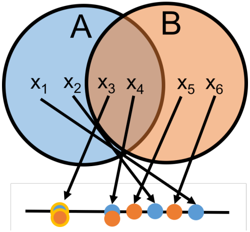

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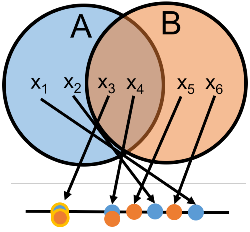

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Locality sensitive: the higher $J(A, B)$ is, the more likely $MinHash(A), MinHash(B)$ are to collide.

Similarity Search with MinHash

Goal: Given a document y, identify all documents x in a database with Jaccard similarity (of their shingle sets) $J(x, y) \geq 1/2$.

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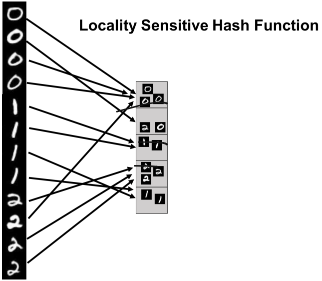
- Create a hash table of size m, choose a random hash function $\mathrm{g} : [0, 1] \rightarrow [m]$, and insert every item x into bucket $\mathrm{g}(MinHash(x))$. Search for items similar to y in bucket $\mathrm{g}(MinHash(y))$.



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- What is $\Pr[\mathbf{g}(MinHash(x)) = \mathbf{g}(MinHash(y))]$ assuming $J(x, y) = 1/2$ and $\mathbf{g}$ is collision free?

$\frac{1}{2} + \delta$ $\approx \frac{1}{m}$

$\frac{1}{m}$

$\frac{1}{2}$

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- For every document x in your database with $J(x, y) \geq 1/2$ what is the probability you will find x in bucket $\mathbf{g}(MinHash(y))$? $\geq 1/2$

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With a simple use of MinHash, we miss a match x with $J(x, y) = 1/2$ with probability $1/2$. How can we reduce this false negative rate?

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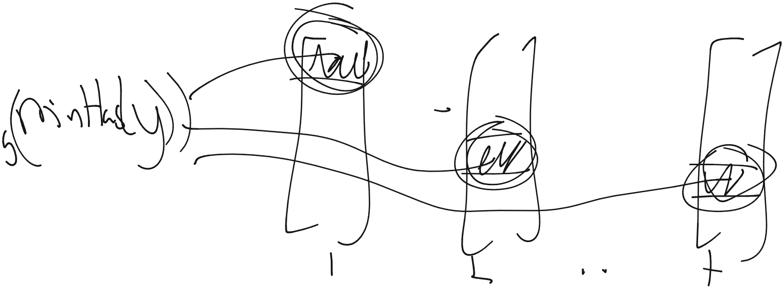

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Reducing False Negatives

With a simple use of MinHash, we miss a match x with $J(x, y) = 1/2$ with probability $1/2$. How can we reduce this false negative rate?

Repetition: Run MinHash t times independently, to produce hash values $MH_1(x), \ldots, MH_t(x)$. Apply random hash function $\mathbf{g}$ to map all these values to locations in t hash tables.

- To search for items similar to y, look at all items in bucket $\mathbf{g}(MH_1(y))$ of the 1^{st} table, bucket $\mathbf{g}(MH_2(y))$ of the 2^{nd} table, etc.
- What is the probability that x with $J(x, y) = 1/2$ is in at least one of these buckets, assuming for simplicity $\mathbf{g}$ has no collisions?
  $1-$ (probability in *no* buckets) $= 1 - \left(\frac{1}{2}\right)^t \approx .99$ for $t = 7$.
- What is the probability that x with $J(x, y) = 1/4$ is in at least one of these buckets, assuming for simplicity $\mathbf{g}$ has no collisions?
  $1-$ (probability in *no* buckets) $= 1 - \left(\frac{3}{4}\right)^t \approx .87$ for $t = 7$.

Potential for a lot of false positives! Slows down search time.

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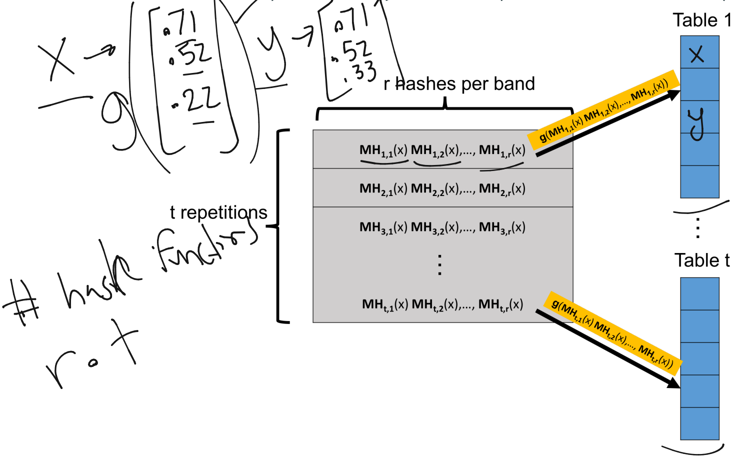

Create t hash tables. Each is indexed into not with a single MinHash value, but with r values, appended together. A length r signature.

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  $x \to \begin{bmatrix} .71 \\ .63 \\ .22 \end{bmatrix}$ $y \to \begin{bmatrix} .71 \\ .63 \\ .22 \end{bmatrix}$

- Probability that x and y having matching signatures in repetition i. $\Pr\left[MH_{i,1}(x), \ldots, MH_{i,r}(x) = MH_{i,1}(y), \ldots, MH_{i,r}(y)\right]$ s^r

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$$\text{Hit Probability: } 1 - (1 - s^r)^t.$$

The s-curve

Using t repetitions each with a signature of r MinHash values, the probability that x and y with Jaccard similarity $J(x, y) = s$ match in at least one repetition is: $1 - (1 - s^r)^t$.

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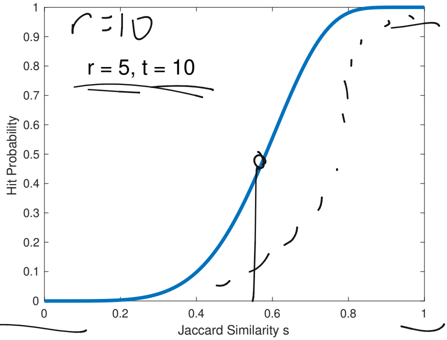

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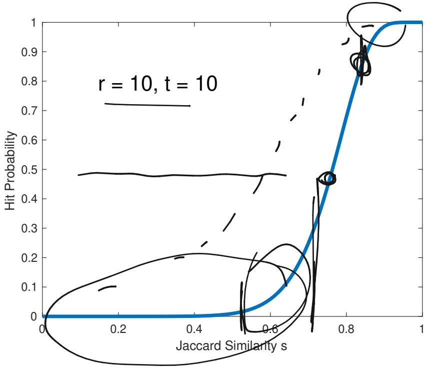

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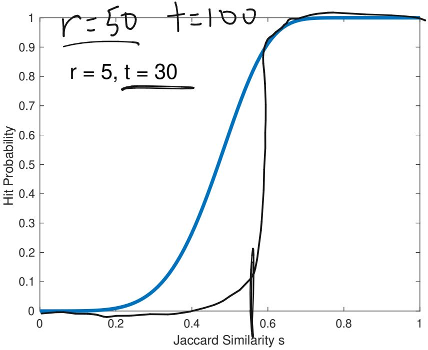

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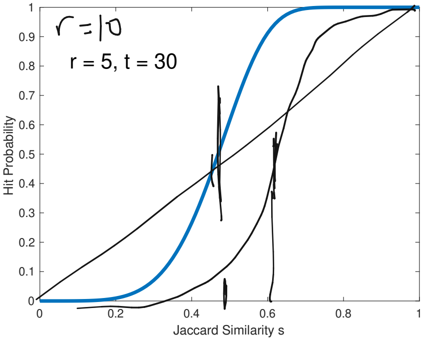

r and t are tuned depending on application. 'Threshold' when hit probability is 1/2 is $\approx (1/t)^{1/r}$. E.g., $\approx (1/30)^{1/5} = .51$ in this case.

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Expected Number of Items Scanned: (proportional to query time)

$$\leq 10 + .98 * 10,000 + .007 * 9,989,990 \approx 80,000$$

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Hashing for Duplicate Detection

	Hash Table	Bloom Filters	MinHash Similarity Search	Distinct Elements
Goal	Check if x is a duplicate of any y in database and return y.	Check if x is a duplicate of y in database.	Check if x is a duplicate of any y in database and return y.	Count # of items, excluding duplicates.
Space	$O(n)$ items	$O(n)$ bits	$O(n \cdot t)$ items (when t tables used)	$O\left(\frac{\log\log n}{\epsilon^2}\right)$
Query Time	$O(1)$	$O(1)$	Potentially o(n)	NA
Approximate Duplicates?	✗	✗	✓	✗

All different variants of detecting duplicates/finding matches in large datasets. An important problem in many contexts!

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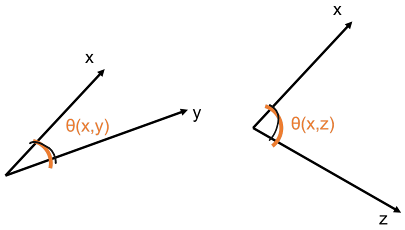

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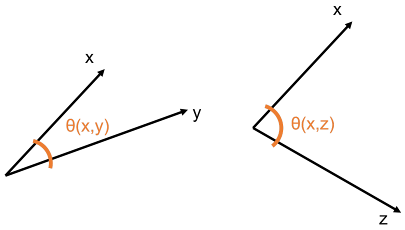

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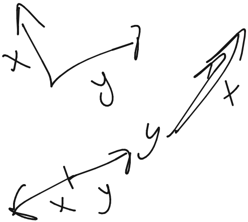

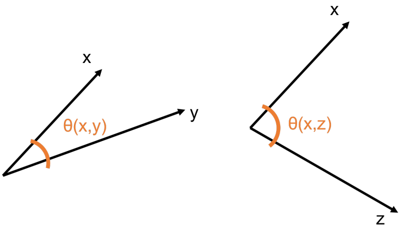

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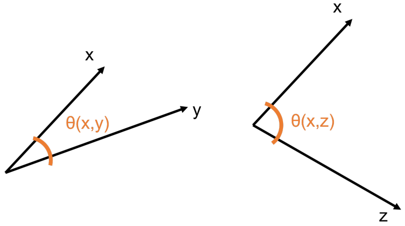

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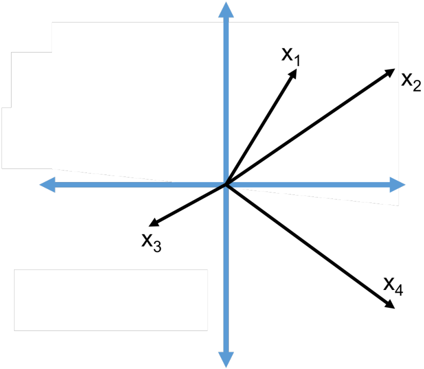

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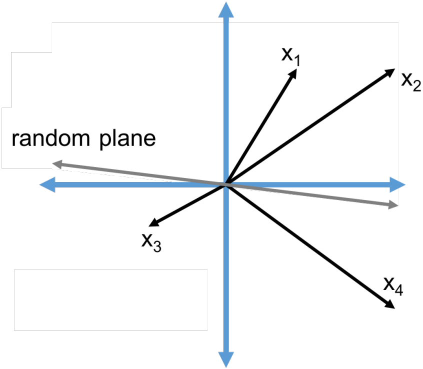

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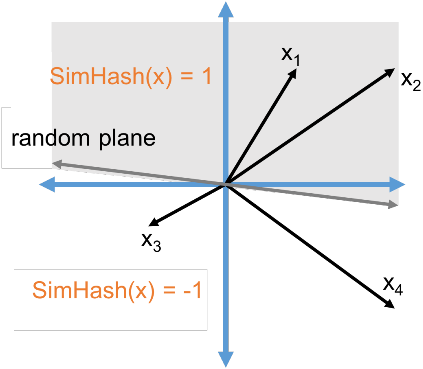

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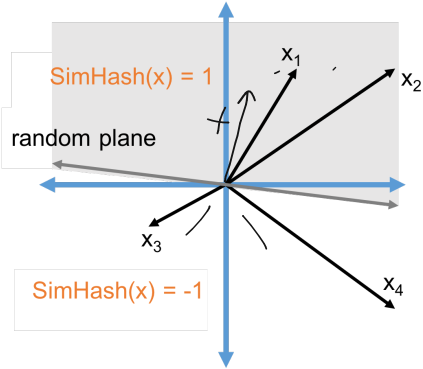

$SimHash(x) = \text{sign}(\langle x, t \rangle)$ for a random vector t.

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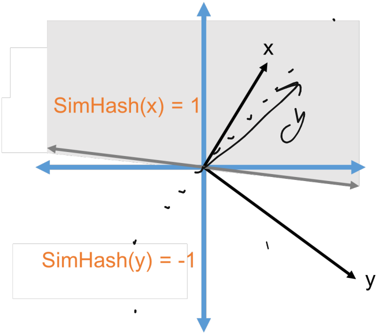

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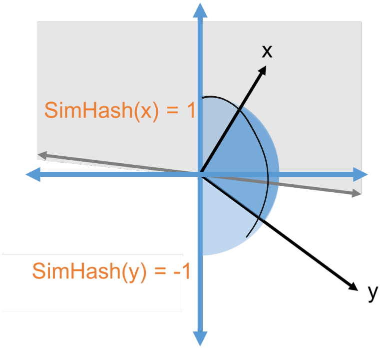

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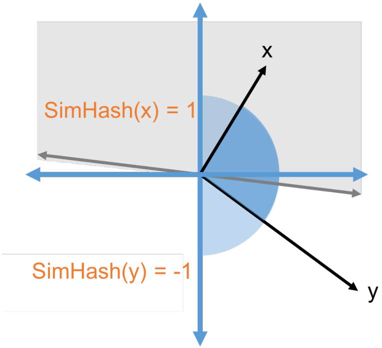

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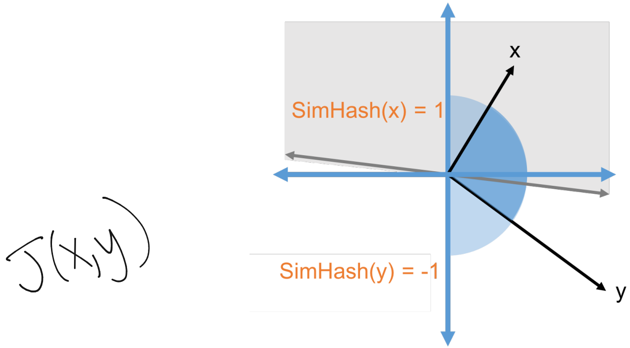

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Questions on MinHash and Locality Sensitive Hashing?