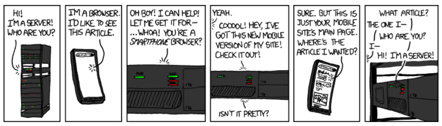
Reasoning about programs



Ways to verify your code

- The hard way:
  - Make up some inputs
  - If it doesn't crash, ship it
  - When it fails in the field, attempt to debug
- The easier way:
  - Reason about possible behavior and desired outcomes
  - Construct simple tests that exercise that behavior
- Another way that can be easy
  - Prove that the system does what you want
    - Rep invariants are preserved
    - Implementation satisfies specification
  - Proof can be formal or informal (we will be informal)
  - Complementary to testing

Reasoning about code

- Determine what facts are true during execution
  - x > 0
  - for all nodes n: n.next.previous == n
  - array a is sorted
  - x + y == z
  - if x != null, then x.a > x.b
- Applications:
  - Ensure code is correct (via reasoning or testing)
  - Understand why code is incorrect

Forward reasoning

- You know what is true before running the code
  What is true after running the code?
- Given a precondition, what is the postcondition?
- Applications:
  Representation invariant holds before running code
  Does it still hold after running code?
- Example:

```
// precondition: x is even
x = x + 3;
y = 2x;
x = 5;
// postcondition:  ??
```

Backward reasoning

- You know what you want to be true after running the code
  What must be true beforehand in order to ensure that?
- Given a postcondition, what is the corresponding precondition?
- Applications:
  (Re-)establish rep invariant at method exit: what's required?
  Reproduce a bug: what must the input have been?
- Example:

```
// precondition:  ??
x = x + 3;
y = 2x;
x = 5;
// postcondition:  y > x
```

- How did you (informally) compute this?

Forward vs. backward reasoning

- Forward reasoning is more intuitive for most people
  - Helps understand what will happen (simulates the code)
  - Introduces facts that may be irrelevant to goal
    Set of current facts may get large
  - Takes longer to realize that the task is hopeless
- Backward reasoning is usually more helpful
  - Helps you understand what should happen
  - Given a specific goal, indicates how to achieve it
  - Given an error, gives a test case that exposes it

Forward reasoning example

```
assert x >= 0;
i = x;
    // x ≥ 0 & i = x
z = 0;
    // x ≥ 0 & i = x & z = 0
while (i != 0) {        ⇐ What property holds here?
  z = z + 1;
  i = i –1;             ⇐ What property holds here?
}
    // x ≥ 0 & i = 0 & z = x
assert x == z;
```

Backward reasoning

Technique for backward reasoning:

- Compute the weakest precondition (wp)
- There is a wp rule for each statement in the programming language
- Weakest precondition yields strongest specification for the computation (analogous to function specifications)

Assignment

```
// precondition: ??
x = e;
// postcondition: Q
```

Precondition: Q with all (free) occurrences of x replaced by e

- Example:

```
// assert: ??
x = x + 1;
// assert x > 0
```

Precondition = (x+1) > 0

Method calls

```
// precondition: ??
x = foo();
// postcondition: Q
```

- If the method has no side effects: just like ordinary assignment
- If it has side effects: an assignment to every variable in modifies

Use the method specification to determine the new value

If statements

```
// precondition: ??
if (b) S1 else S2
// postcondition: Q
```

Essentially case analysis:

wp("if (b) S1 else S2", Q) =

(b ⇒ wp("S1", Q)

⋀ ¬ b ⇒ wp("S2", Q))

If: an example

```
// precondition: ??
if (x == 0) {
  x = x + 1;
} else {
  x = (x/x);
}
// postcondition: x ≥ 0
```

Precondition:

wp("if (x==0) {x = x+1} else {x = x/x}", x ≥ 0) =
= (x = 0 ⇒ wp("x = x+1", x ≥ 0)
  & x ≠ 0 ⇒ wp("x = x/x", x ≥ 0))
= (x = 0 ⇒ x + 1 ≥ 0) & (x ≠ 0 ⇒ x/x ≥ 0)
= 1 ≥ 0 & 1 ≥ 0
= true

Reasoning About Loops

- A loop represents an unknown number of paths
  - Case analysis is problematic
  - Recursion presents the same issue
- Cannot enumerate all paths
  - That is what makes testing and reasoning hard

Loops: values and termination

```
// assert x ≥ 0 & y = 0
while (x != y) {
    y = y + 1;
}
// assert x = y
```

1) Pre-assertion guarantees that x ≥ y
2) Every time through loop
   x ≥ y holds and, if body is entered, x > y
   y is incremented by 1
   x is unchanged
   Therefore, y is closer to x (but x ≥ y still holds)
3) Since there are only a finite number of integers between x and y, y will eventually equal x
4) Execution exits the loop as soon as x = y

Understanding loops by induction

- We just made an inductive argument
  Inducting over the number of iterations
- Computation induction
  Show that conjecture holds if zero iterations
  Assume it holds after n iterations and show it holds after n+1
- There are two things to prove:
  Some property is preserved (known as "partial correctness")
  loop invariant is preserved by each iteration
  The loop completes (known as "termination")
  The "decrementing function" is reduced by each iteration

Loop invariant for the example

```
// assert x ≥ 0 & y = 0
while (x != y) {
    y = y + 1;
}
// assert x = y
```

- So, what is a suitable invariant?
- What makes the loop work?
  LI = x ≥ y

1) $x \geq 0 \;\&\; y = 0 \Rightarrow LI$
2) $LI \;\&\; x \neq y$ {`y = y+1;`} LI
3) $(LI \;\&\; \neg(x \neq y)) \Rightarrow x = y$

Is anything missing?

```
// assert x ≥ 0 & y = 0
while (x != y) {
    y = y + 1;
}
// assert x = y
```

Does the loop terminate?

Total Correctness via Well-Ordered Sets

- We have not established that the loop terminates
- Suppose that the loop always reduces some variable's value. Does the loop terminate if the variable is a
  - Natural number?
  - Integer?
  - Non-negative real number?
  - Boolean?
  - ArrayList?
- The loop terminates if the variable values are (a subset of) a well-ordered set
  - Ordered set
  - Every non-empty subset has least element

Decrementing Function

- Decrementing function D(X)
  - Maps state (program variables) to some well-ordered set
  - This greatly simplifies reasoning about termination
- Consider: `while (b) S;`
- We seek D(X), where X is the state, such that
  1. An execution of the loop reduces the function's value: LI & b {S} $D(X_{post}) < D(X_{pre})$
  2. If the function's value is minimal, the loop terminates: (LI & D(X) = minVal) ⇒ ¬b

Proving Termination

```
// assert x ≥ 0 & y = 0
// Loop invariant: x ≥ y
// Loop decrements: (x-y)
while (x != y) {
    y = y + 1;
}
// assert x = y
```

- Is "x-y" a good decrementing function?

1. Does the loop reduce the decrementing function's value?
   // assert (y ≠ x); let $d_{pre} = (x – y)$
   y = y + 1;
   // assert $(x_{post} – y_{post}) < d_{pre}$
2. If the function has minimum value, does the loop exit?
   (x ≥ y & x – y = 0) ⇒ (x = y)

Choosing Loop Invariant

- For straight-line code, the wp (weakest precondition) function gives us the appropriate property
- For loops, you have to guess:
  - The loop invariant
  - The decrementing function
- Then, use reasoning techniques to prove the goal property
- If the proof doesn't work:
  - Maybe you chose a bad invariant or decrementing function
    - Choose another and try again
  - Maybe the loop is incorrect
    - Fix the code
- Automatically choosing loop invariants is a research topic

In practice

I don't routinely write loop invariants

I do write them when I am unsure about a loop and when I have evidence that a loop is not working

- Add invariant and decrementing function if missing
- Write code to check them
- Understand why the code doesn't work
- Reason to ensure that no similar bugs remain

More on Induction

- Induction is a very powerful tool

$$2^n = 1 + \sum_{k=1}^{n} 2^{k-1}$$

Proof by induction:

For n=1, $1 + \sum_{k=1}^{1} 2^{k-1} = 1 + 2^0 = 1 + 1 = 2 = 2^1$

Inductive step

Assume $2^m = 1 + \sum_{k=1}^{m} 2^{k-1}$ and show that $2^{m+1} = 1 + \sum_{k=1}^{m+1} 2^{k-1}$

$$2^{m+1} = 1 + \sum_{k=1}^{m+1} 2^{k-1} = 1 + \sum_{k=1}^{m} 2^{k-1} + 2^m = 2^m + 2^m = 2 \times 2^m = 2^{m+1}$$

Is Induction Too Powerful?