(1) (Fisher Information) Let \( f(x; \theta) = \theta e^{-\theta x}, x \geq 0 \) be the probability distribution of a random variable \( X \).
- Compute the score function for \( f \).
- Compute the Fisher information for \( f, \theta \). What is the Cramer-Rao bound on the unbiased estimator?
- Suppose we observe iid samples of \( X \), namely \( X_1, \ldots, X_n \). What would be the Cramer-Rao bound on the unbiased estimator of \( \theta \) from this \( n \) samples?
- Design an estimator for \( \theta \) from \( X_1, \ldots, X_n \). Does your estimator match the Cramer-Rao bound?

(2) (Channel Capacity) Consider the channel where the input alphabet \( \mathcal{X} = \{0, 1\} \), output alphabet \( \mathcal{Y} = \{0, 1, a, 1 + a\} \). The input-output transition probabilities are given by,

\[
p(y|x) = \begin{cases} 
\frac{1}{2}, & \text{if } y = x; \\
\frac{1}{2}, & \text{if } y = x + a; \\
0, & \text{otherwise}.
\end{cases}
\]

What is the capacity of this channel?

(3) (Channel Capacity) Consider the channel where the input and output alphabets are same, \( \mathcal{X} = \mathcal{Y} = \{0, 1\} \). The channel transition probabilities are given by,

\[
p(y|x) = \begin{cases} 
1, & \text{if } y = x = 0; \\
p, & \text{if } x = 1, y = 0; \\
1 - p, & \text{if } x = 1, y = 1; \\
0, & \text{otherwise}.
\end{cases}
\]

What is the capacity of this channel?

(4) Consider a commander of an army besieged in a fort for whom the only means of communication to his allies is a set of carrier pigeons. Assume that each carrier pigeon can carry one letter (8 bits), that pigeons are released once every 5 minutes, and that each pigeon takes exactly 3 minutes to reach its destination.
- Assuming that all the pigeons reach safely, what is the capacity of this link in bits/hour?
- Now assume that the enemies try to shoot down the pigeons and that they manage to hit a fraction \( \alpha \) of them. Since the pigeons are sent at a constant rate, the receiver knows when the pigeons are missing. What is the capacity of this link?
• Now assume that the enemy is more cunning and that every time they shoot down a pigeon, they send out a dummy pigeon carrying a random letter (chosen uniformly from all 8-bit letters). What is the capacity of this link in bits/hour?

Set up an appropriate model for the channel in each of the above cases, and indicate how to go about finding the capacity.

(5) (Exponential noise channels.) Let the output $Y_i$ and input $X_i$ of a channel is related by $Y_i = X_i + Z_i$, where $Z_i$ is i.i.d. exponentially distributed noise with mean $\mu$. Assume that we have a mean constraint on the signal (i.e., $EX_i \leq \lambda$). What is the capacity of such a channel?

(6) Consider the binary Hamming code of length 15, i.e., the code whose parity-check matrix is formed all the 15 nonzero columns of length 4, taken in the lexicographic order (from 0001 to 1111).
• What is the dimension $k$ and distance $d$ of the code (explain your answers).
• Write out a generator matrix of the code such that the message bits are bits $1, 2, \ldots, k$.
• You are given a received vector $z = 000000 * *0000111$ where $*$ stands for erasure. Perform a decoding of $z$ with the code. What is/are the candidate codeword(s)? Explain your answer.

(7) Consider the following Generator matrix of a code:

$$
G = \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

• What is the length ($n$) and dimension ($k$) of the code? How many codewords are there?
• Write down a parity-check matrix for the code.
• What is the minimum distance of this code? How many errors can this code correct? How many erasures can this code correct?
• Decode the following received word: 111000, to a valid codeword.