# COMPSCI 311: Introduction to Algorithms

## Lecture 25: Randomized/Approximation Algorithms and Review

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## Coping With NP-Completeness

Suppose you want to solve an NP-complete problem?
What should you do?

You can't design an algorithm to do *all* of the following:

1. Solve arbitrary instances of the problem
2. Solve problem to optimality
3. Solve problem in polynomial time

Coping strategies

1. Design algorithms for special cases of problem.
2. Design approximation algorithms or heuristics.
3. Design algorithms that run efficiently for some, but not all, problem instances

Another frontier: randomized algorithms

## Approximation Algorithms

- **Def**: $\rho$-approximation algorithm
  - Runs in polynomial time
  - Solves arbitrary instances of the problem
  - Guaranteed to find a solution within ratio $\rho$ of optimum:

$$\frac{\text{value of our solution}}{\text{value of optimum solution}} \geq \rho \qquad \text{(if goal = maximum)}$$

## Randomization + Approximation: Max-3-Sat

Max-3-Sat. Given a 3-Sat formula, find a truth assignment that satisfies as many clauses as possible
(Note: three *distinct* variables per clause)

$$\begin{aligned}
C_1 &= x_2 \vee \bar{x}_3 \vee \bar{x}_4 \\
C_2 &= x_2 \vee x_3 \vee \bar{x}_4 \\
C_3 &= \bar{x}_1 \vee x_2 \vee x_4 \\
C_4 &= \bar{x}_1 \vee \bar{x}_2 \vee x_3 \\
C_5 &= x_1 \vee \bar{x}_2 \vee \bar{x}_4
\end{aligned}$$

**Remark** NP-*hard* search problem

**Simple idea**. Set each variable true with probability $\frac{1}{2}$, independently

## Randomized Max-3-Sat

For any clause $C_i$:

$$\Pr[\text{don't satisfy } C_i] = (\tfrac{1}{2})^3 = \tfrac{1}{8}$$
$$\Pr[\text{satisfy } C_i] = \tfrac{7}{8}$$

Assume $k$ clauses $\implies$ expected number of satsified clauses $= \frac{7}{8}k$
(linearity of expectation)

**Corollary**: **expected** number of clauses satisfied by a random assignment is $\geq \frac{7}{8}$ of optimal (no assignment can satisfy more than $k$)

A **randomized approximation** algorithm (guarantee for **expected** value)

## Clicker

Consider a 2-Sat instance with $k$ clauses where each clause has two distinct variables. Suppose each variable is set to true independently with probability $\frac{1}{2}$. What is the expected number of satisfied clauses?

A. $\frac{1}{4}k$

B. $\frac{1}{2}k$

C. $\frac{3}{4}k$

D. $\frac{7}{8}k$

## Probabilistic Method

Prove an object exists by showing that a randomized procedure finds it with nonzero probability.

**Corollary**: For every 3-Sat instance with $k$ clauses, there is a truth assignment that satisfies $\geq \frac{7}{8}k$ clauses.

**Proof**: Expected number of satisfied clauses is $\frac{7}{8}k$;
a random variable is *at least* expected value with nonzero probability

**Corollary**. Every 3-SAT instance with $\leq 7$ clauses is satisfiable!

**Proof**: There is some assignment that satisfies $\geq \frac{7}{8}k$ clauses. Then

$$\#\text{ unsatisfied clauses} \leq \frac{k}{8} \leq \frac{7}{8} < 1$$

There are no unsatisfied clauses.

## Clicker

For what number of clauses can we guarantee that a 2-Sat formula is satisfiable?

A. 2 or fewer

B. 3 or fewer

C. 3 or more

D. 4 or fewer

Example:

$$(x_1 \vee x_2) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2)$$

## What if We Want to *Guarantee* to Satisfy $\frac{7}{8}k$ clauses?

**Fact**. We can **derandomize** the previous algorithm $\Rightarrow$ deterministic poly-time algorithm to satisfy $\geq \frac{7}{8}k$ clauses.

- Idea: try $x_1 = 1$ and $x_1 = 0$, calculate expected $\#$ satisfied clauses. Choose best and repeat.

**Fact**. No poly-time algorithm can find an assignment satisfying $\geq (\frac{7}{8} + \epsilon)k$ for every satisfiable formula unless P $=$ NP.

## Summary

Final Topics

- (No randomized, approximation algorithms)
- Polynomial-time reductions
- NP-completeness
- Network flows
- Dynamic programming
- Divide-and-conquer
- MST
- Greedy
- Graph algorithms and definitions
- Asymptotic analysis

## Reminder

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## Polynomial Time Reductions

- We focus on decision problems, e.g., for input $\langle G, k\rangle$, does there exist a vertex cover with at most $k$ nodes?
- Given two decision problems $X$ and $Y$, $X \leq_P Y$ means that it's possible to transform an input $I$ of $X$ into an input $f(I)$ of $Y$ in polynomial time such that

  $$I \text{ is a yes instance of } X \text{ iff } f(I) \text{ is a yes instance of } Y$$

  The transformation is a reduction from $X$ to $Y$.
- We saw examples such as

  $$\textsc{VertexCover} \leq_P \textsc{IndependentSet}$$

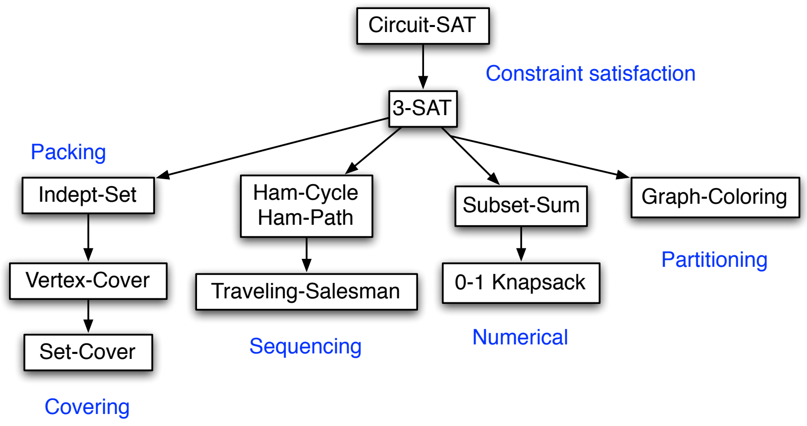
  $$\text{3-SAT} \leq_P \textsc{IndependentSet}$$

- Useful property: If $X \leq_P Y$ and $Y \leq_P Z$ then $X \leq_P Z$.

## NP Completeness

- $P$ = set of problems you can solve in polynomial time, e.g., minimum spanning tree, matchings, flows, shortest path.
- $NP$ = set of problems you can verify in polynomial time:
  - If the answer should be yes then there's some extra input (a "witness" or "certificate" or "hint") that you can be given that makes it easy (i.e., in poly time) to check answer is yes
  - If the answer should be "no" then there is no such input.
- $Y$ is $NP$-Complete if $Y \in NP$ and $X \leq_P Y \; \forall \; X \in NP$.
- Useful Properties: Suppose $X \leq_P Y$. Then
  - If $Y \in P$ then $X \in P$.
  - If $Y \in NP$ and $X$ is NP-complete then $Y$ is also NP complete
  - If $Y \in P$ and $X$ is NP-complete then $P = NP$.
- $NP$-Complete problems are in some sense the hardest problems in NP. If you can solve one of them in polynomial time then you prove $P = NP$. But very few people believe this is possible.

## NP Complete Problems



Circuit-SAT

Constraint satisfaction

3-SAT

Packing

Indept-Set

Ham-Cycle
Ham-Path

Subset-Sum

Graph-Coloring

Vertex-Cover

Traveling-Salesman

0-1 Knapsack

Partitioning

Set-Cover

Sequencing

Numerical

Covering

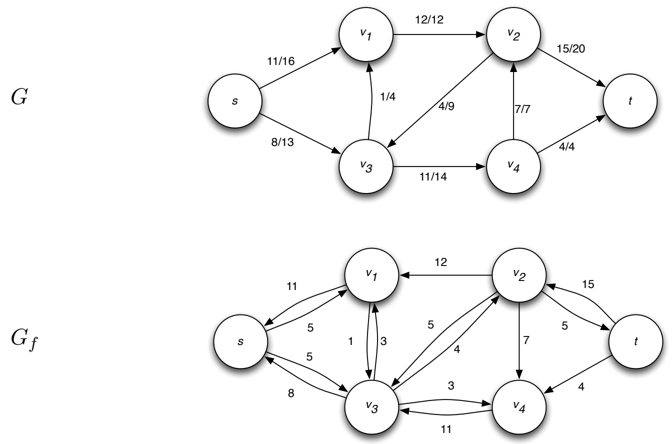
## Network Flows

- Flow network
  - Directed graph
  - Source node $s$ and target node $t$
  - Edge capacities $c(e) \geq 0$
- Flow
  - Capacity Constraints: $0 \leq f(e) \leq c(e)$ on each edge
  - Flow conservation: for all $v \notin \{s, t\}$,

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

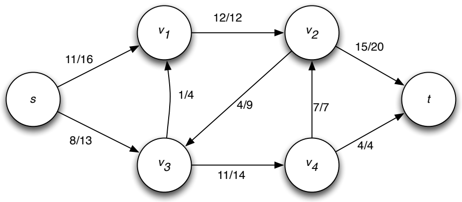
  - Value $v(f)$ of flow $f$ = total flow on edges leaving source
  - **Max flow problem**: find a flow of maximum value
- Residual network encodes how you can change the current flow without violating the capacity constraints.
- Ford Fulkerson Algorithm: Repeatedly increases the flow by finding augmenting paths in the residual network.

## Augment Example

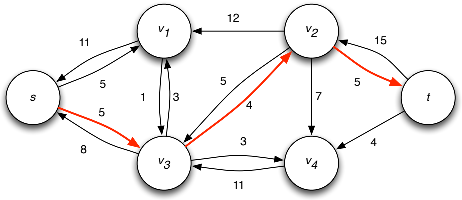


## Augmenting Path

$G$

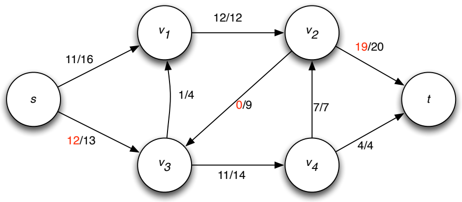


$G_f$

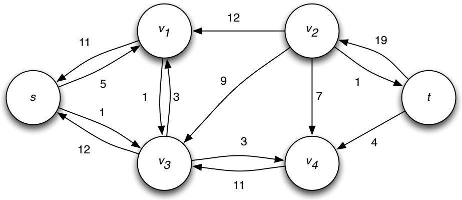


## New Flow

$G$



$G_f$



## Max-Flow Min-Cut Theorem

- $v(f) \leq c(A, B)$ for any flow $f$ and any $s$-$t$ cut $c(A, B)$
- Upon termination, Ford-Fulkerson produces a flow $f$ and cut $(A, B)$ such that $v(f) = c(A, B)$, so $f$ is a max-flow and $(A, B)$ is a min-cut
- The cut $(A, B)$ is found by letting $A =$ set of nodes reachable from $s$ in residual graph

## Dynamic Programming

- Design technique based on recursion. Identify recursive structure by writing recurrence for optimal value.
- The recurrence identifies all subproblems.
- Solve them in a systematic way starting from simplest ones first (base case)

**Example**: Weighted interval scheduling

- $\text{OPT}(j) = \max\{\text{OPT}(j-1), w_j + \text{OPT}(p(j))\}$
- $\text{OPT}(0) = 0$
- Compute $\text{OPT}(j)$ iteratively for $j = 0$ to $n$
- Running time $O(n)$

## Divide-And-Conquer

- Design technique:
  - Often: divide input into equal sized chunks, solve each recursively, combine to solve original problem
  - Can be more subtle—e.g., integer multiplication
  - Tip: don't think about what happens inside recursion. "Magic"
- Solving recurrences, e.g., $T(n) \leq 2T(n/2) + O(n)$
  - Recursion tree, unrolling
  - "Guess and verify": proof by induction
  - Master theorem Suppose $T(n) = aT(n/b) + O(n^d)$. Then:

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

## MST

- Definitions: spanning tree, MST, cut
- Cut property: lightest edge across any cut belongs to every MST
- Prim's algorithm: maintain a set $S$ of explored nodes. Add cheapest edge from $S$ to $V - S$. Repeat.
- Kruskal's algorithm: consider edges in order of cost. Add edge if it does not create a cycle.
- Cycle property: most expensive edge in any cycle does not belong to MST

## Greedy Algorithms

- Greedy algorithms are "short sighted" algorithms that take each step based on what looks good in the short term.
  - **Example:** Kruskal's Algorithm adds lightest edge that doesn't complete a cycle when building an MST.
  - **Example:** When maximizing the number of non-overlapping TV shows we always added the show that finished earliest out of the remaining shows.

## Greedy Algorithms

- Things to note:
  - If a greedy algorithm requires first sorting the input, remember to include the running time of sorting in your overall analysis.
  - Focus on correctness proofs: "greedy stays ahead", "exchange argument", induction, contradiction
- What to know:
  - Apply/adapt proof techniques for scheduling problems; solve similar problems
  - Working knowledge of MST algorithms, Dijkstra: apply to concrete examples, understand principles and proof techniques

## Graph Algorithms: BFS and DFS Trees

- BFS from node $s$:
  - Partitions nodes into layers $L_0 = \{s\}, L_1, L_2, L_3 \ldots$
  - $L_i$ defined as neighbors of nodes in $L_{i-1}$ that aren't already in $L_0 \cup L_1 \cup \ldots \cup L_{i-1}$.
  - $L_i$ is set of nodes at distance exactly $i$ from $s$
  - Returns tree $T$: for any edge $(u, v)$ in graph, $u$ and $v$ are in same layer or adjacent layer
  - Can be used to test whether $G$ is bipartite, find shortest path from $s$ to $t$
- DFS from node $s$
  - Returns DFS tree $T$ rooted at $s$
  - For any edge $(u, v)$, $u$ is an ancestor of $v$ in the tree or vice versa.
- Both run in time $O(m + n)$
- Both can be used to find connected components of graph, test whether there is a path from $s$ to $t$

## Related "Traversal" Algorithms

Algorithms that grow a set $S$ of explored nodes from starting node $s$

- BFS (traversal): add all nodes $v$ that are neighbors of some node $u \in S$. Repeat.
- Dijkstra (shortest paths): add node $v$ with smallest value of $d(u) + \ell(u, v)$ for some node $u$ in $S$, where $d(u)$ is distance from $s$ to $u$. Repeat.
- Prim (MST): add node $v$ with smallest value of $c(u, v)$ where $u \in S$. Repeat.

## Bipartite, Directed Graphs

- An undirected graph $G$ is bipartite if its nodes can be colored red and blue such that no edge has two endpoints of the same color
  - $G$ is bipartite if and only if it does not contain an odd cycle
  - $G$ is bipartite if and only if, after running BFS from any node, there is no edge between two nodes in the same layer
- A directed graph is acyclic (a DAG) if there is no directed cycle
  - There is no directed cycle if and only if there is a topological ordering.
  - Can find a topological order using the fact that a DAG has a node with no incoming edges.

## Asymptotic Analysis

Given two positive functions $f(n)$ and $g(n)$:

- $f(n)$ is $O(g(n))$
  - iff $\exists c > 0, n_0 \geq 0$ s.t. $f(n) \leq cg(n)$ for all $n \geq n_0$
- $f(n)$ is $\Omega(g(n))$
  - iff $\exists c > 0, n_0 \geq 0$ s.t. $f(n) \geq cg(n)$ for all $n \geq n_0$
  - iff $g(n)$ is $O(f(n))$
- $f(n)$ is $\Theta(g(n))$
  - iff $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$
- Know how to apply definitions, compare functions, use to analyze running time of algorithms

## Thanks!

Good luck on the final, and please fill out the course survey!

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