# COMPSCI 311: Introduction to Algorithms

## Lecture 24: More NP-Complete Problems

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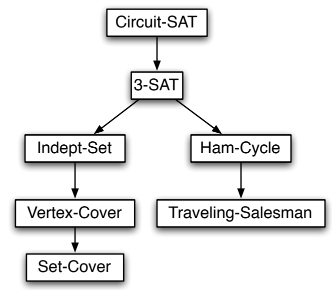
## NP-Complete Problems So Far




Arrows show reductions discussed in class.
We could construct a polynomial reduction between any pair.

## NP-Completeness and Reductions

**Careful, direction of reduction matters!**

A $\leq_P$ B: A reduces **to** B (A "no harder" than B)
From arbitrary instance of A, construct instance of B
Reduction and construction is **one-way**

Problem instances are **equivalent** (**both ways**):
$\text{YES}_A \implies \text{YES}_B$
$\text{YES}_B \implies \text{YES}_A$ (same as $\text{NO}_A \implies \text{NO}_B$)

B is NP-complete means:

1. B is in NP: can **check** solution in polynomial time ("easy enough")
2. B is NP-hard: some NP-complete A reduces **to** B: A $\leq_P$ B ("hard enough"). We also say: reduce **from** A.

## Clicker

Which of the following graph problems are in NP?

A. Length of longest simple path is $\leq k$

B. Length of longest simple path is $= k$

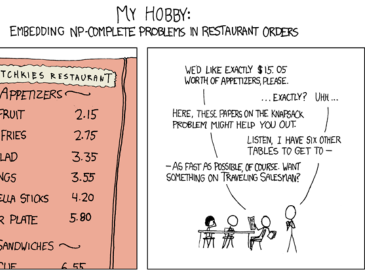
C. Length of longest simple path is $\geq k$

D. Find length of longest simple path.

E. All of the above.

## Numerical problems

**Subset Sum** decision problem: given $n$ items with weights $w_1, \ldots, w_n$, is there a subset of items whose weight is exactly $W$?




Dynamic programming: $O(nW)$ *pseudo-polynomial* time algorithm (not polynomial in input length $n \log W$)

## Subset Sum Warmup

Does this instance have a solution?

```
w1   1010
w2   1001
w3   0110
w4   0101
     ----
 W   1111
```

A. Yes
B. No

## Subset Sum Warmup

For which nonzero values of y does this instance have a solution?

```
10010
10001
01001
01010
00111
00100
-----
1113y
```

A. $y = 1$
B. $y = 1, 2$
C. $y = 1, 2, 3$

## Subset Sum Warmup

For which nonzero values of y does this instance have a solution?

```
10010
10011
01001
01000
00111
00100
-----
1112y
```

A. $y = 1$
B. $y = 1, 2$
C. $y = 1, 2, 3$

## Subset Sum

**Theorem.** Subset sum is NP-complete.

Reduction from $3\text{-SAT}$. ($n$ variables, $m$ clauses).

## Subset Sum Reduction

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

| | variable digits | | | clause digits |
|---|---|---|---|---|
| Item | 1 | 2 | 3 | |
| $t_1$ | 1 | 0 | 0 | |
| $f_1$ | 1 | 0 | 0 | |
| $t_2$ | 0 | 1 | 0 | |
| $f_2$ | 0 | 1 | 0 | |
| $t_3$ | 0 | 0 | 1 | |
| $f_3$ | 0 | 0 | 1 | |
| $W$ | 1 | 1 | 1 | |

- Items $t_i, f_i$ for each $x_i$; correspond to truth assignment
- Weights $\implies$ select exactly one
- (Numbers are base 10)

## Subset Sum Reduction

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

| | variable digits | | | clause digits | | |
|---|---|---|---|---|---|---|
| Item | 1 | 2 | 3 | 4 | 5 | 6 |
| $t_1$ | 1 | 0 | 0 | 1 | 0 | 0 |
| $f_1$ | 1 | 0 | 0 | 0 | 1 | 1 |
| $t_2$ | 0 | 1 | 0 | 0 | 1 | 0 |
| $f_2$ | 0 | 1 | 0 | 1 | 0 | 1 |
| $t_3$ | 0 | 0 | 1 | 1 | 0 | 1 |
| $f_3$ | 0 | 0 | 1 | 0 | 1 | 0 |
| $W$ | 1 | 1 | 1 | ? | ? | ? |

- Clause digit equal to 1 iff $x_i$ assignment satisfies $C_j$
- Total for clause digit $> 0$ iff assignment satisfies $C_j$
- Goal: all clause digits $> 0$. How to set $W$ to enforce this? Total could be 1, 2, 3 for satisfied clause.

## Subset Sum Reduction

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

| | variable digits | | | clause digits | | |
|---|---|---|---|---|---|---|
| Item | 1 | 2 | 3 | 4 | 5 | 6 |
| $t_1$ | 1 | 0 | 0 | 1 | 0 | 0 |
| $f_1$ | 1 | 0 | 0 | 0 | 1 | 1 |
| $t_2$ | 0 | 1 | 0 | 0 | 1 | 0 |
| $f_2$ | 0 | 1 | 0 | 1 | 0 | 1 |
| $t_3$ | 0 | 0 | 1 | 1 | 0 | 1 |
| $f_3$ | 0 | 0 | 1 | 0 | 1 | 0 |
| $W$ | 1 | 1 | 1 | 3 | 3 | 3 |

- Set all clause digits of $W$ to 3... then add dummy items to increase total by **at most two**.

## Subset Sum Reduction

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

| | variable digits | | | clause digits | | |
|---|---|---|---|---|---|---|
| Item | 1 | 2 | 3 | 4 | 5 | 6 |
| $t_1$ | 1 | 0 | 0 | 1 | 0 | 0 |
| $f_1$ | 1 | 0 | 0 | 0 | 1 | 1 |
| $t_2$ | 0 | 1 | 0 | 0 | 1 | 0 |
| $f_2$ | 0 | 1 | 0 | 1 | 0 | 1 |
| $t_3$ | 0 | 0 | 1 | 1 | 0 | 1 |
| $f_3$ | 0 | 0 | 1 | 0 | 1 | 0 |
| $W$ | 1 | 1 | 1 | 3 | 3 | 3 |

| | variable digits | | | clause digits | | |
|---|---|---|---|---|---|---|
| Item | 1 | 2 | 3 | 4 | 5 | 6 |
| $y_1$ | 0 | 0 | 0 | 1 | 0 | 0 |
| $z_1$ | 0 | 0 | 0 | 1 | 0 | 0 |
| $y_2$ | 0 | 0 | 0 | 0 | 1 | 0 |
| $z_2$ | 0 | 0 | 0 | 0 | 1 | 0 |
| $y_3$ | 0 | 0 | 0 | 0 | 0 | 1 |
| $z_3$ | 0 | 0 | 0 | 0 | 0 | 1 |

- **Two** dummy items per clause $\Rightarrow$ can increase total by up to 2
- Can make total exactly 3 iff total of non-dummy items is $> 0$

## Subset Sum Reduction: Details (Review on Own)

- All weights have $n + m$ digits
- For variable $x_i$, create two items $t_i, f_i$
  - Both have $i$th digit equal to 1
  - All other items have zero in this digit
  - $i$th digit of $W = 1 \Rightarrow$ select exactly one of $t_i, f_i$
- The $n + j$th digit corresponds to clause $C_j$
  - If $x_i \in C_j$, set $n + j$th digit of $t_i = 1$
  - If $\neg x_i \in C_j$, set $n + j$th digit of $f_i = 1$
  - Everything else $0$.

- Set $n + j$th digit of $W = 3$
  - Consider a subset of items corresponding to a truth assignment (exactly one of $t_i, f_i$)
  - If $C_j$ is not satisfied, then total in position $n + j$ is 0, otherwise it is 1, 2, or 3
  - Create two "dummy" items $y_j, z_j$ with $1$ in position $n + j$
  - Can select dummies to yield total of 3 in position $n + j$ iff $C_j$ is satisfied

## Subset Sum Proof

- All numbers (including $W$) are polynomially long.
- If $\Phi$ satisfiable,
  - Select $t_i$ if $x_i = 1$ in satisfying assignment else select $f_i$.
  - Take $y_j, z_j$ as needed.
- If subset exists with sum $W$
  - Either $t_i$ or $f_i$ is chosen. Assign $x_i$ accordingly.
  - For each clause, at least one term must be selected, otherwise clause digit is $< 3$.

## Graph Coloring

**Def.** A $k$-coloring of a graph $G = (V, E)$ is a function $f : V \to \{1, \ldots, k\}$ such that for all $(u, v) \in E$, $f(u) \neq f(v)$.

**Problem.** Given $G = (V, E)$ and number $k$, does $G$ have a $k$-coloring?

Many applications

- Actually coloring maps!
- Scheduling jobs on machine with competing resources.
- Allocating variables to registers in a compiler.

**Claim.** $2$-COLORING $\in$ P (equivalent to bipartite testing)

**Theorem.** $3$-COLORING is NP-Complete.

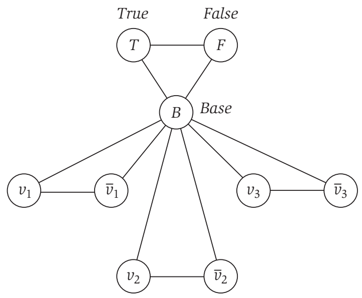
## 3-Color: Gadget for Variables

- Reduce from $3$-SAT.

3 colors: True, False, "Base"

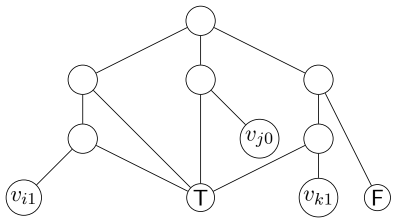
3 special nodes in a clique $T, F, B$.
For each variable $x_i$, two nodes $v_{i0}, v_{i1}$.
Edges $(v_{i0}, B), (v_{i1}, B), (v_{i0}, v_{i1})$.

Either $v_{i0}$ or $v_{i1}$ colored $T$, the other colored $F$.




## Reduction: Clause Gadget

For clause $x_i \vee \neg x_j \vee x_k$




Top node can be colored iff not all three $v$-nodes are $F$.

## Proof

- Graph is polynomial in $n + m$.
- If satisfying assignment
  - Color $B, T, F$ then $v_{i1}$ as $T$ if $\phi(x_i) = 1$.
  - Since clauses satisfied, can color each gadget.
- If graph 3-colorable
  - One of $v_{i0}, v_{i1}$ must get $T$ color.
  - Clause gadget colorable iff clause satisfied.

**Question.** What about $k$-coloring?

## Clicker Question

Which of the following is true?

A. If we can reduce 3-coloring to $k$-coloring, then $k$-coloring is NP-complete

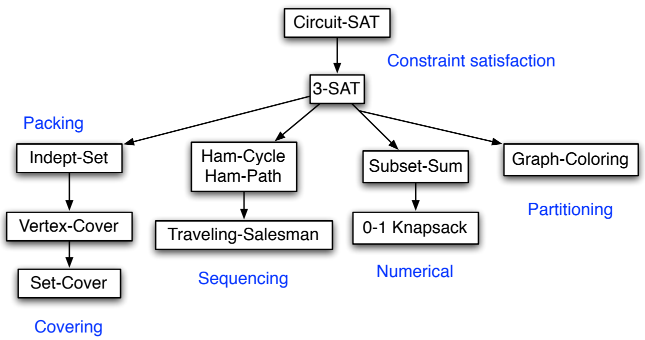
B. $k$-coloring is NP-complete since any 3-coloring is also a $k$-coloring for $k \geq 3$

C. $k$-coloring is not NP-complete since 3-coloring is the hardest case, for $k > 3$ the coloring is easier

D. $k$-coloring is not NP-complete because the 4-color theorem has been proved

## NP-Completeness Recap

Types of hard problems:




. . . any many others. See book or other sources for more examples. You can use *any known NP-complete* problem to prove a new problem is NP-complete.