COMPSCI 311: Introduction to Algorithms

Lecture 13: Closest Pair of Points

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Finding Minimum Distance between Points

- **Problem 1**: Given n points on a line $p_1, p_2, \dots, p_n \in \mathbb{R}$, find the closest pair: $\min_{i \neq j} |p_i - p_j|$.
  - Compare all pairs $O(n^2)$
  - Better algorithm? Sort and compare adjacent pairs. $O(n \log n)$
- **Problem 2:** Now what if the points are in $\mathbb{R}^2$?
  - Compare all pairs $O(n^2)$
  - Sort? Points can be close in one coordinate and far in other
  - We'll do it in $O(n \log n)$ steps using divide-and-conquer.

Problem Formulation

- **Input**: set of points $P = \{p_1, \dots, p_n\}$ where $p_i = (x_i, y_i)$
- **Assumption**: we can iterate over points in order of x- or y- coordinate in $O(n)$ time. Pre-generate data structures to support this in $O(n \log n)$ time.

Minimum Distance: Recursive Algorithm

1. Find vertical line L to split points into sets P_L, P_R of size $n/2$. $O(n)$
2. Recursively find minimum distance in P_L and P_R.
   - δ_L = minimum distance between $p, q \in P_L, p \neq q$. $T(n/2)$
   - δ_R = same for P_R. $T(n/2)$
3. δ_M = minimum distance between $p \in P_L, q \in P_R$. ??
4. Return $\min(\delta_L, \delta_R, \delta_M)$.

Naive Step 3 takes $\Omega(n^2)$ time. But if we do it in $O(n)$ time we get

$$T(n) = 2T(n/2) + O(n) \implies T(n) = O(n \log n)$$

Making Step 3 Efficient

- **Goal**: given δ_L, δ_R, compute $\min(\delta_L, \delta_R, \delta_M)$
- Let $\delta = \min(\delta_L, \delta_R)$. If $p \in P_L, q \in P_R$ are at least δ apart, they cannot be a closer pair, so we can ignore pair (p, q).
- Let S be the set of points within distance δ from L. We only need to consider pairs that are both in S.
- For a given point $p \in S$, how many other points in S are within δ units of p in the y coordinate? **Intuition**: point in S on either side of line can't be too close to one another $\implies$ must "spread out" vertically

How to find closest pair with one point in each side?

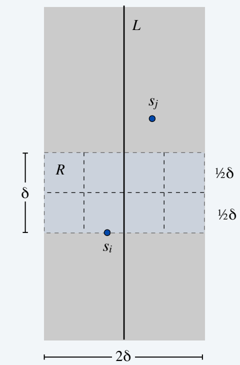
Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|j - i| > 7$, then the distance between s_i and s_j is at least δ.

Pf.
- Consider the 2δ-by-δ rectangle R in strip whose min y-coordinate is y-coordinate of s_i.
- Distance between s_i and any point s_j above R is $\geq \delta$.
- Subdivide R into 8 squares.
- At most 1 point per square.
- At most 7 other points can be in R. ▪

diameter is $\delta / \sqrt{2} < \delta$

constant can be improved with more refined geometric packing argument



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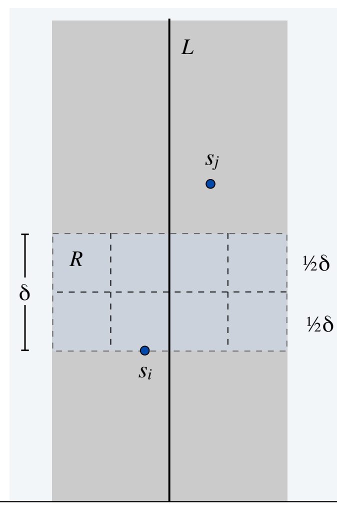
What is the maximum number of points with larger y coordinate that we need to compare to s_i?

A. 7 points
B. 8 points
C. 4 points
D. 0 points



Wrap-Up

- Step 3 is $O(n)$: iterate in order of y coordinate and compare each point to constant number of neighbors.
- $\implies O(n \log n)$ overall.
- **Intuition**: we reduced Step 3 (almost) to 1D closest-pair
  - Iterate, compare each point to next k points (instead of 1)
  - The set S is "nearly one-dimensional". Points cannot be packed too tightly, because pairs on each side have to be at least δ apart.
- For $d > 2$ dimensions, there is a divide and conquer algorithm where the "combine" step (i.e., Step 3) solves a closest pair problem in $d - 1$ dimensions

Closest Pair in d Dimensions

Board work

Solve recurrence

$$T(n, d) = 2T(n/2, d) + T(n, d-1)$$

Base case $T(n, 2) = \Theta(n \log n)$

Solution: $T(n, d) = \Theta(n \log^{d-1} n)$