Island Hopping and Path Colouring

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Optical Network Design

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- Many interesting theory problems arise...

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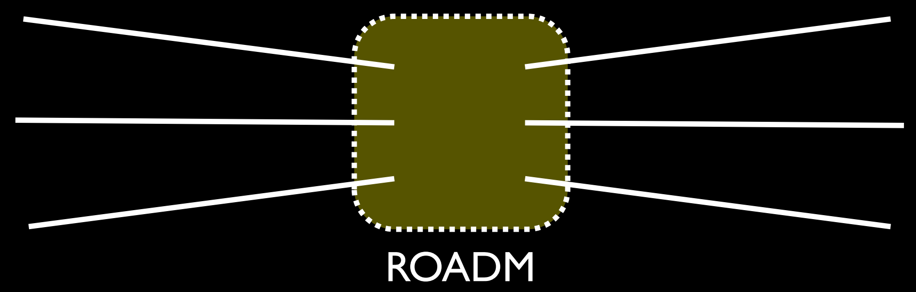

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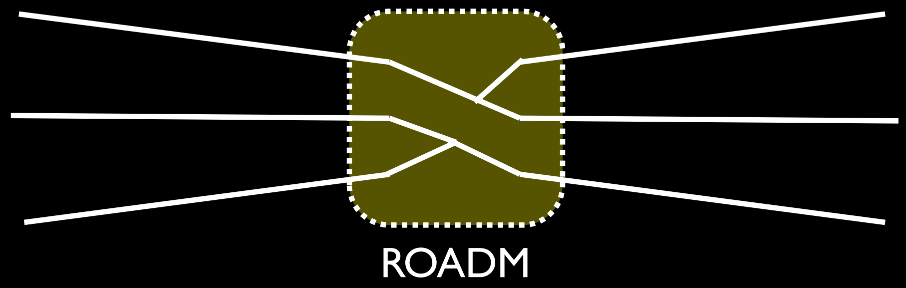

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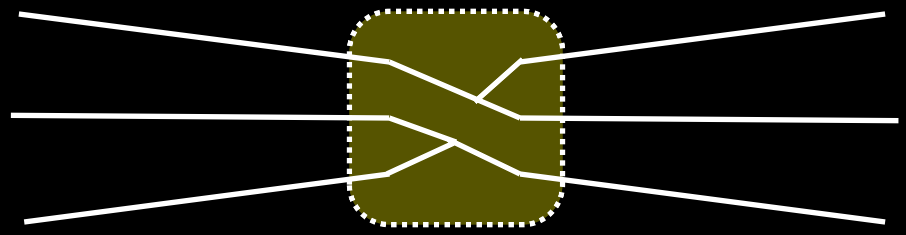

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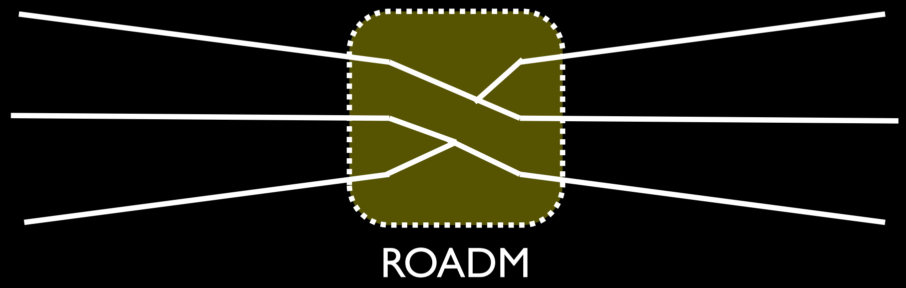

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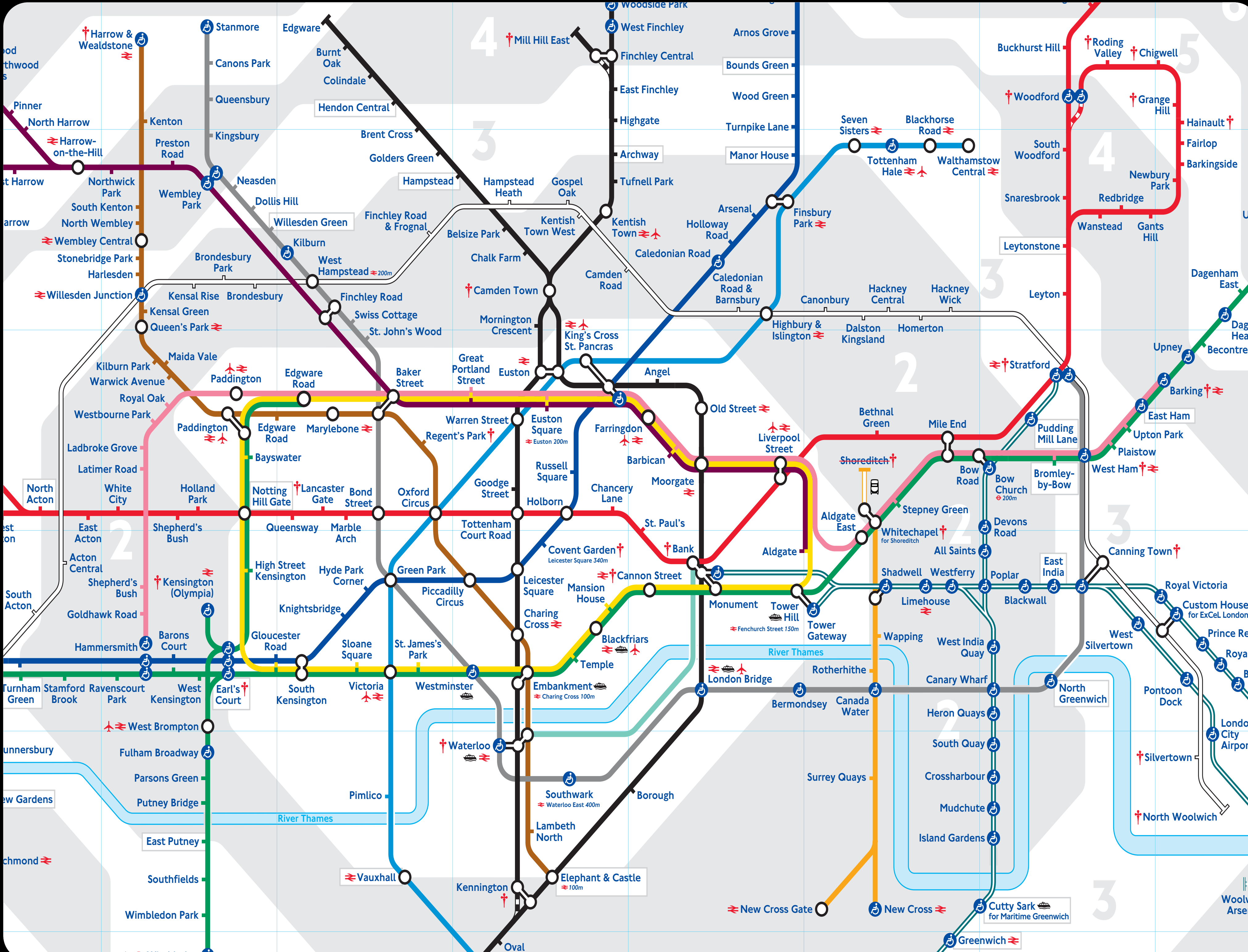
Harrow & Wealdstone
Stanmore
Edgware
Burnt Oak
Colindale
Hendon Central
Brent Cross
Golders Green
Hampstead
Mill Hill East
Woodside Park
West Finchley
Finchley Central
East Finchley
Highgate
Archway
Tufnell Park
Kentish Town
Arnos Grove
Bounds Green
Wood Green
Turnpike Lane
Manor House
Seven Sisters
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Becontree
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Upton Park
Plaistow
West Ham
Pinner
North Harrow
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Kenton
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Queensbury
Canons Park
Kingsbury
Northwick Park
Wembley Park
South Kenton
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Stonebridge Park
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Willesden Junction
Kensal Green
Queen's Park
Kilburn Park
Maida Vale
Warwick Avenue
Royal Oak
Westbourne Park
Ladbroke Grove
Latimer Road
North Acton
White City
East Acton
Holland Park
Shepherd's Bush
Acton Central
South Acton
Shepherd's Bush
Kensington (Olympia)
Goldhawk Road
Hammersmith
Barons Court
Turnham Green
Stamford Brook
Ravenscourt Park
West Kensington
Earl's Court
West Brompton
Fulham Broadway
Parsons Green
Putney Bridge
East Putney
Southfields
Wimbledon Park
Wimbledon
Kew Gardens
Richmond
Neasden
Dollis Hill
Willesden Green
Kilburn
Brondesbury Park
Kensal Rise
Brondesbury
West Hampstead
Finchley Road
Swiss Cottage
St. John's Wood
Finchley Road & Frognal
Hampstead Heath
Gospel Oak
Belsize Park
Kentish Town West
Chalk Farm
Camden Town
Camden Road
Caledonian Road
Caledonian Road & Barnsbury
Holloway Road
Arsenal
Finsbury Park
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Canonbury
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Homerton
Mornington Crescent
King's Cross St. Pancras
Euston
Great Portland Street
Baker Street
Edgware Road
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Regent's Park
Angel
Old Street
Farringdon
Barbican
Moorgate
Liverpool Street
Bethnal Green
Mile End
Stratford
Pudding Mill Lane
Bow Road
Bow Church
Bromley-by-Bow
Shoreditch
Aldgate East
Whitechapel
Stepney Green
Aldgate
Bayswater
Notting Hill Gate
Lancaster Gate
Bond Street
Oxford Circus
Goodge Street
Russell Square
Holborn
Chancery Lane
St. Paul's
Bank
Tottenham Court Road
Queensway
Marble Arch
Covent Garden
Leicester Square
Cannon Street
Mansion House
Monument
Tower Hill
Tower Gateway
High Street Kensington
Hyde Park Corner
Green Park
Piccadilly Circus
Charing Cross
Knightsbridge
Gloucester Road
Sloane Square
St. James's Park
Westminster
Embankment
Temple
Blackfriars
South Kensington
Victoria
Waterloo
Southwark
Lambeth North
Pimlico
Vauxhall
Kennington
Elephant & Castle
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Bermondsey
Canada Water
Rotherhithe
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Shadwell
Limehouse
Westferry
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All Saints
Devons Road
East India
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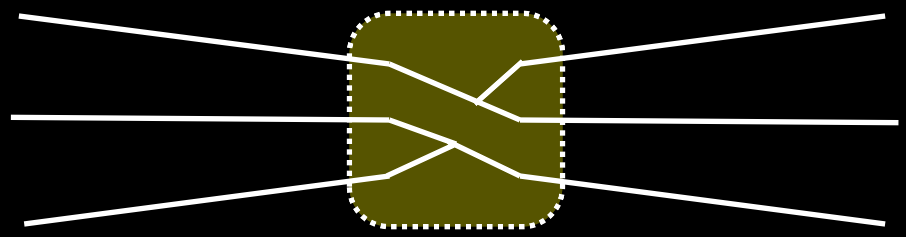

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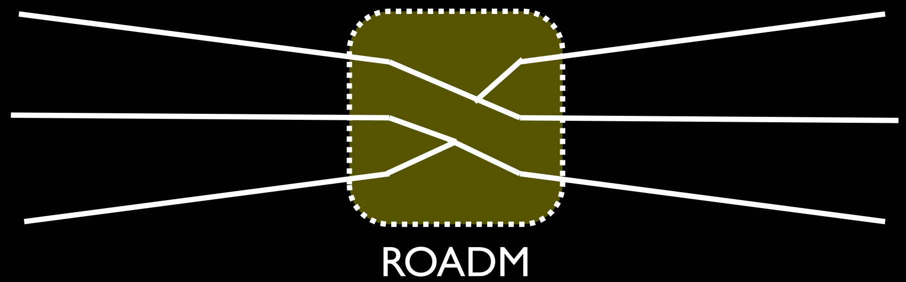

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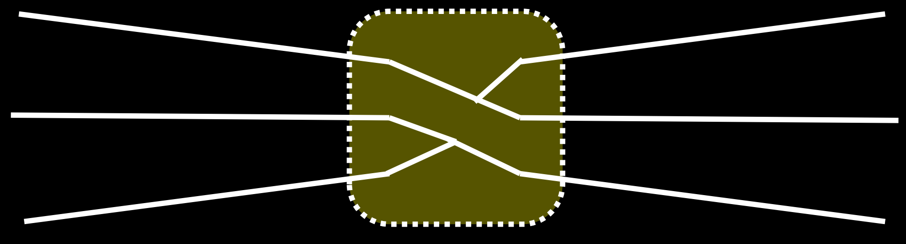

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1. Min-Hop
2. Min-Fiber
3. Min-Both?

MinHop_c

- *Input:* Supply network $G=(V,E)$ and demands H.
- *Solution:*

  a) Decomposition of E into "transparent islands"

  b) Simple routing path P_h for each demand h
- *Goal:* Minimize average number of times each P_h needs to hop between transparent islands.

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Undirected Graphs & 2-arm Roadms

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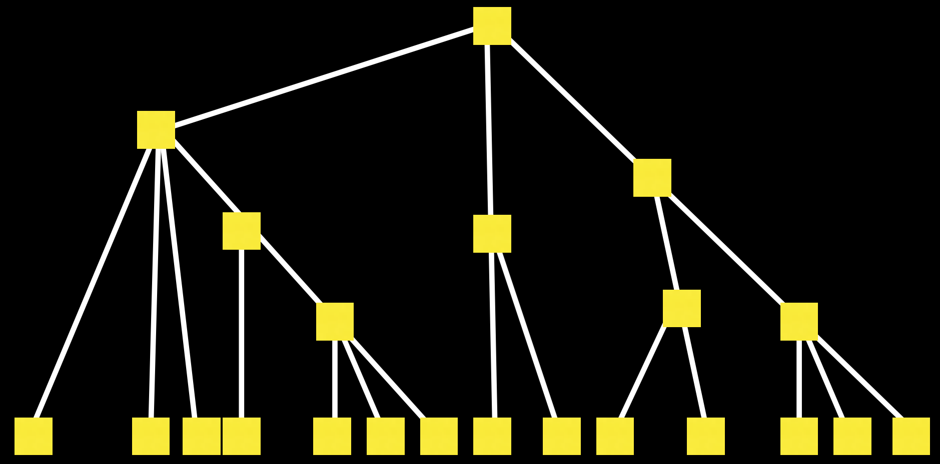

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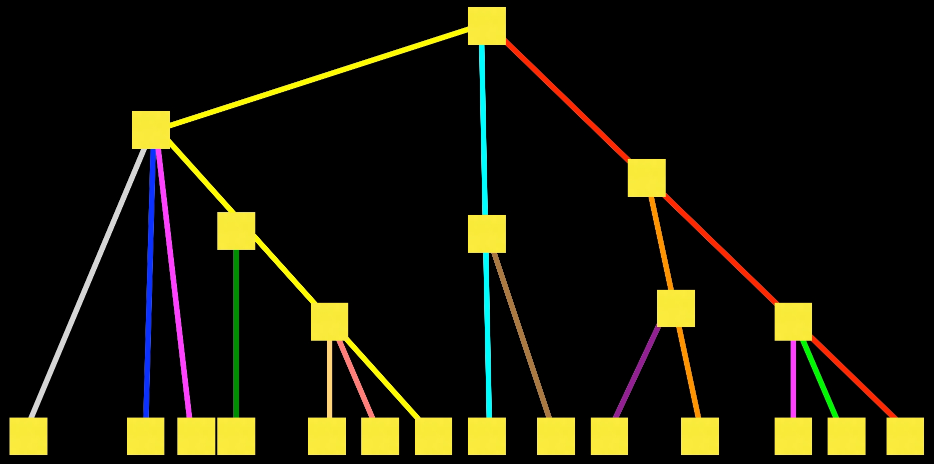

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  Replace each node u with $K_{2,3} = \{u_1, u_2, v_1, v_2, v_3\}$ and match v_1, v_2, v_3 to neighbours of u.

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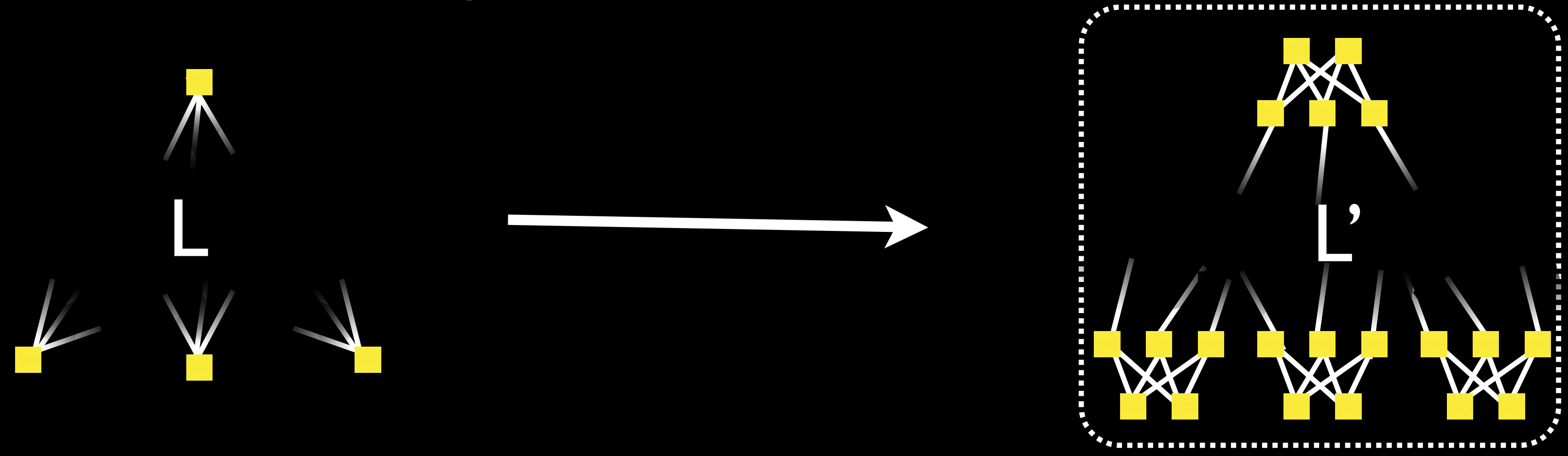

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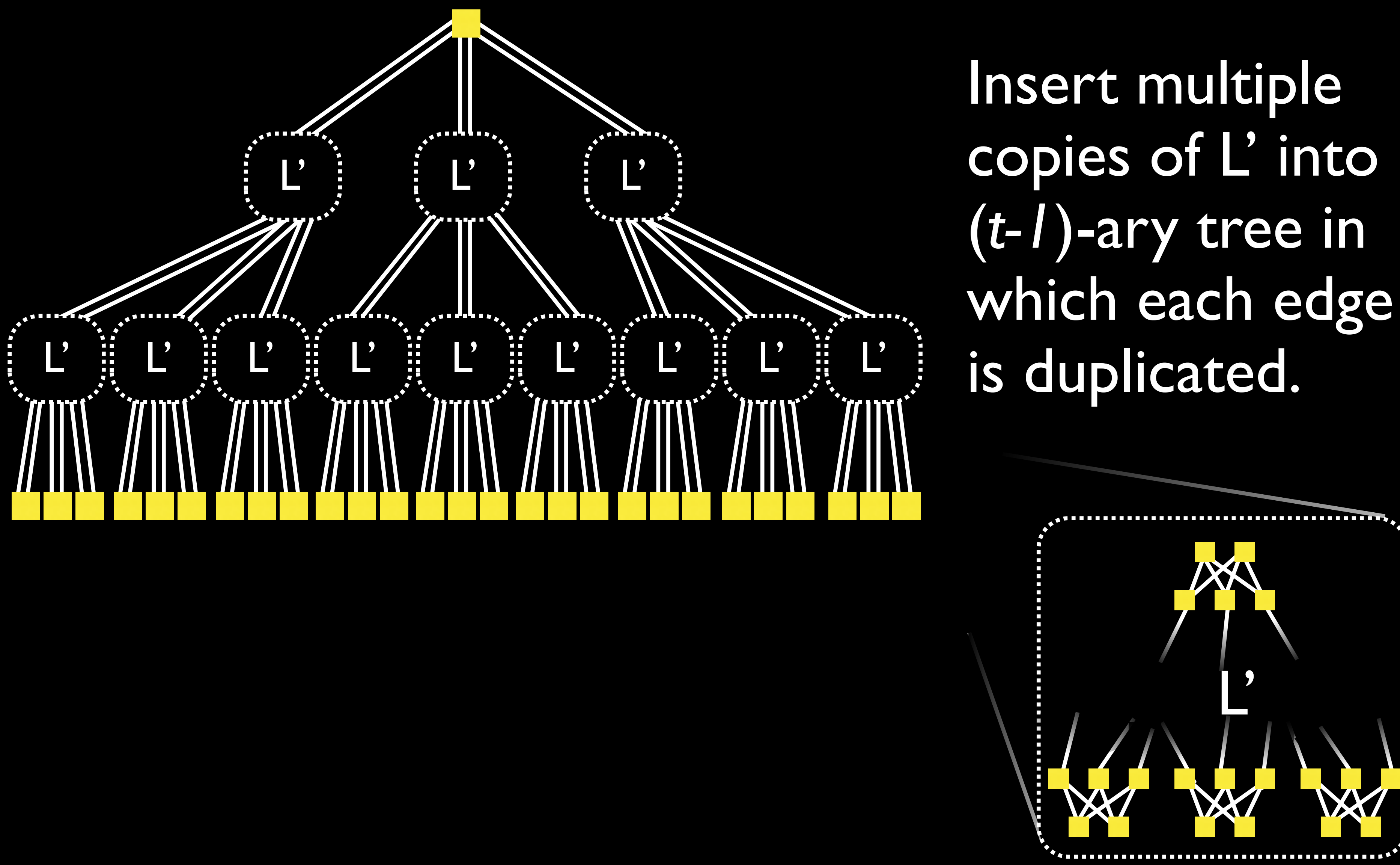

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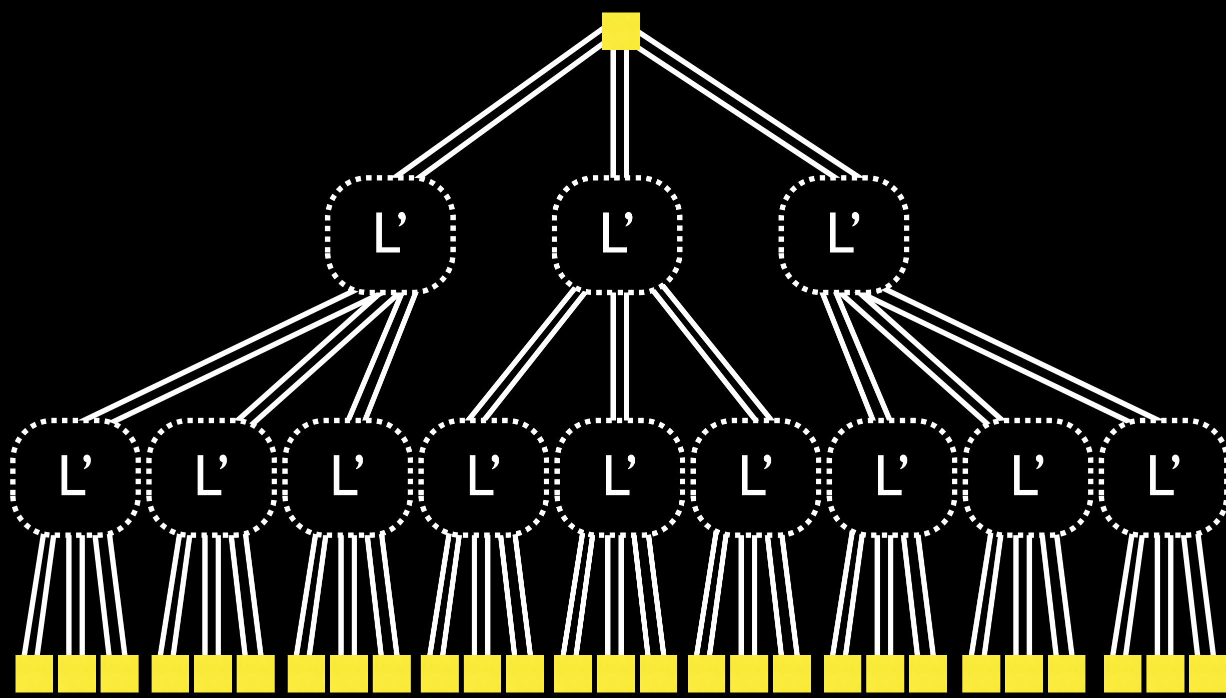

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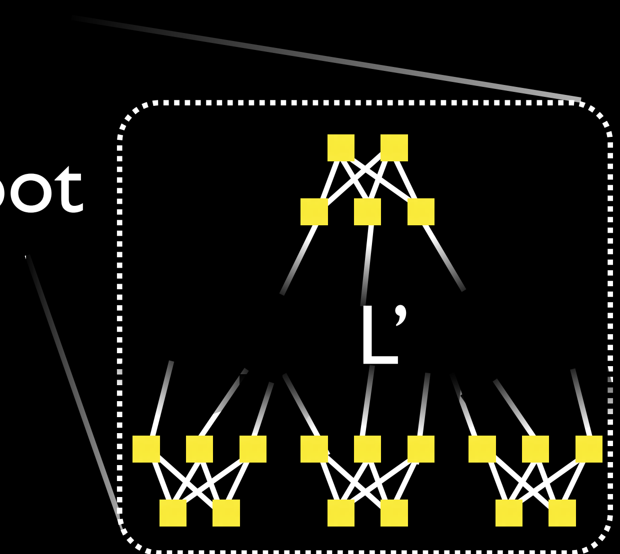

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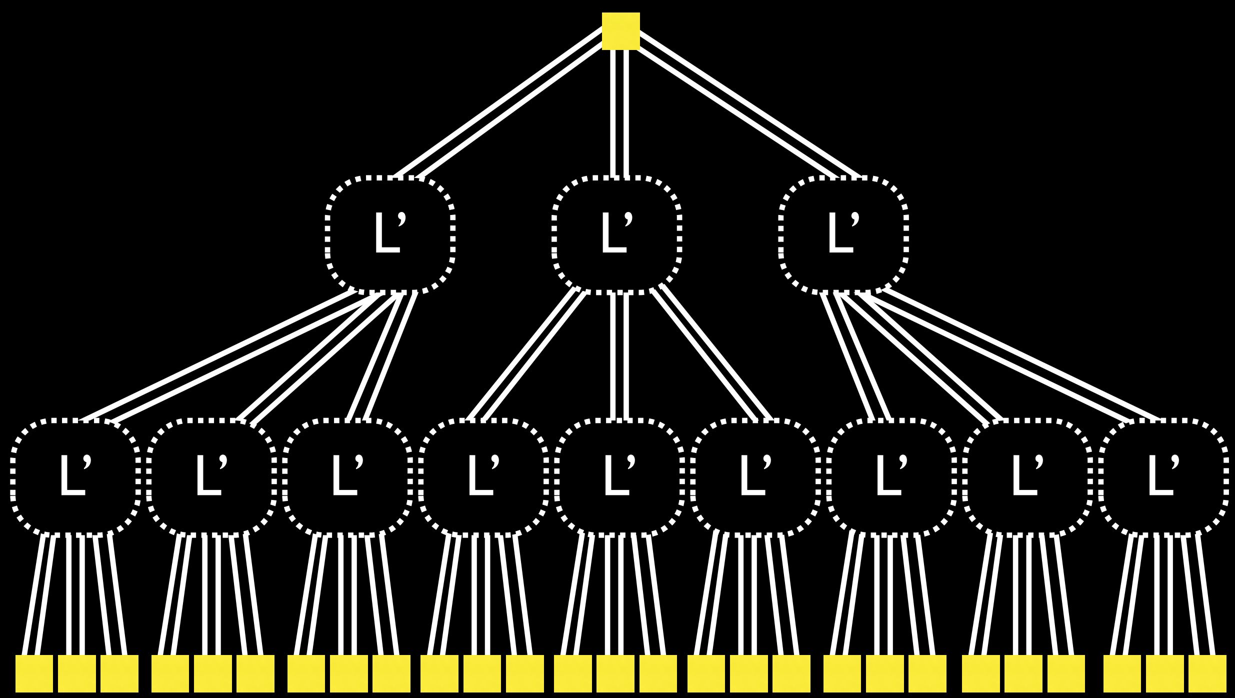

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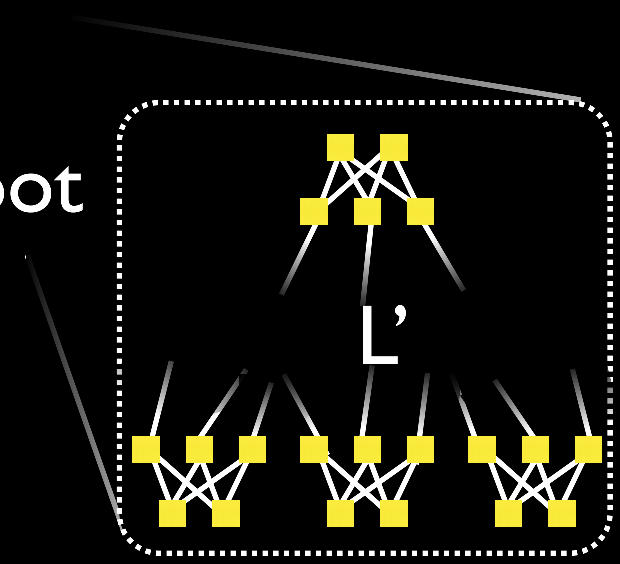

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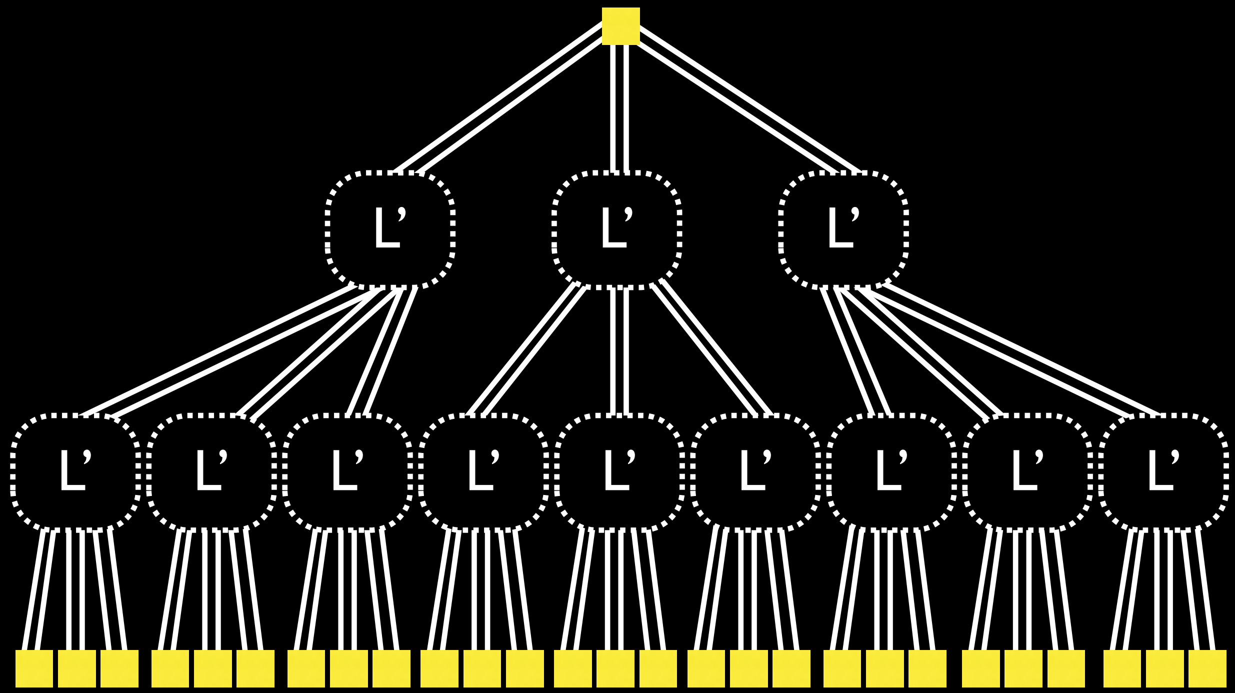

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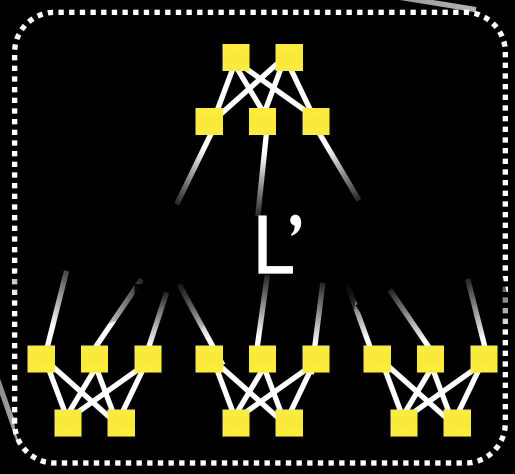

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Finding a solution of cost $o(\log^{1-\epsilon} n)$ requires finding length $\Omega(t)$ path in L.

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  For directed graph L and s_1,t_1,s_2,t_2, it is NP-hard to determine if there is edge disjoint paths between s_1 and t_1; and s_2 and t_2.

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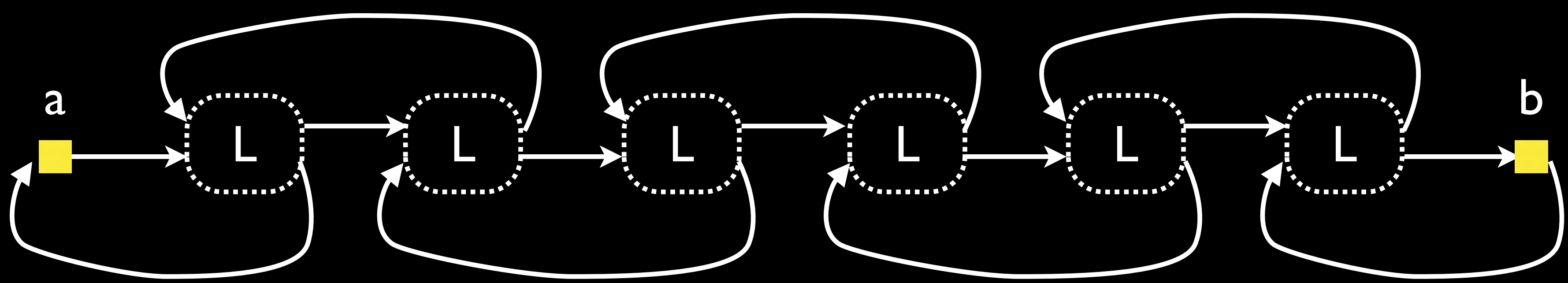

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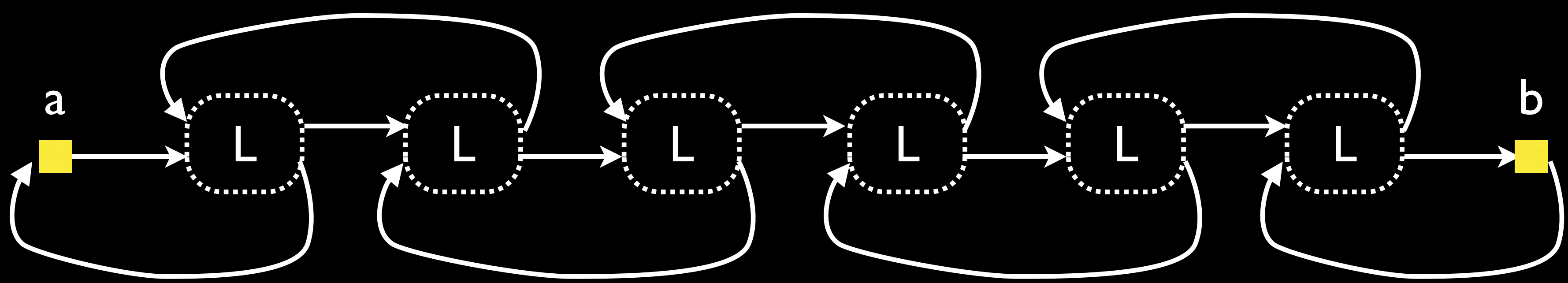

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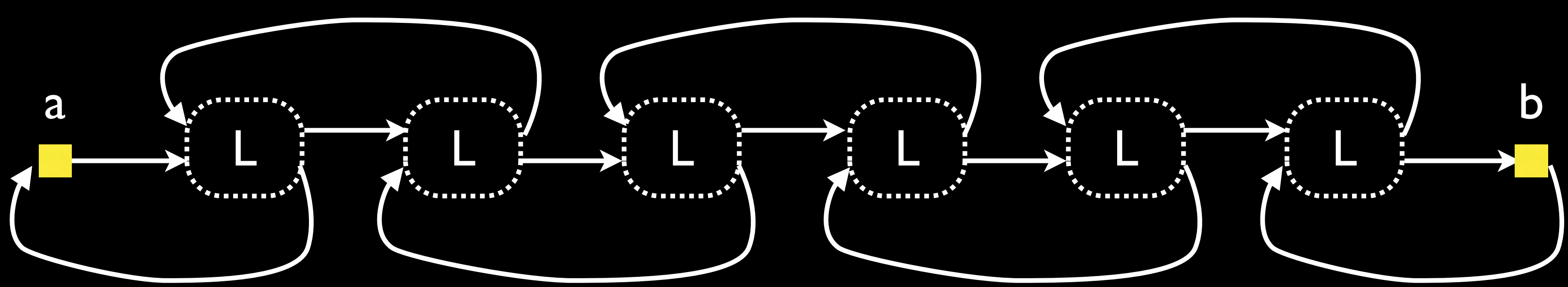

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- Can assume G is strongly connected...

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  Proof: Induction on k: $k=1$ trivial!

  Let q be minimum repeated element and let sequence be of the form $S\ q\ I_1\ q\ I_2\ q\ \ldots\ I_j\ q\ P$.

  Assume $I_1\ I_2\ \ldots\ I_j$ has length r and so $I_1\ q\ I_2\ q\ \ldots\ I_j\ q$ has length at most 2r.

  Sequence $S\ q\ P$ is boosted and has alphabet size k-r hence length is 2(k-r) by induction.

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  Proof (Sketch):

    Define "long" paths $P_1, P_2, ..., P_k$ that route some demands

      P_j: If shortest route in G augmented by edges of distance n^{-2} between all pairs of nodes in P_i for all $i<j$ is length at least $n^{1/2}$ then let P_j be this route.

    Define transparent islands as maximal sub-paths of $P_j \backslash (P_1, ... P_{j-1})$ and all remaining individual edges.

    G is a DAG implies that k=O($n^{1/2}$)

    Boosting lemma implies every routing requires O(k) hops

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  $MinHop_2(G)=1$ if *G* is 4-node connected.

  Graph is Hamiltonian [Tutte '56] *and a degree-3 spanning tree can be found in polytime* [Fürer, Raghavachari '56].

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Summary of MinHop

	2-arm Roadms		*3-arm Roadms*
	Algorithm	*Hardness*	*Algorithm*
Undirected	$O(\log n)$	$\Omega(\log^{1-\epsilon} n)$	$O(\log n)$
Strongly Connected	$O(n)$	$\Omega(n^{1-\epsilon})$	$O(\log n)$
DAG	$O(n^{1/2})$	$\Omega(\log n)$	$O(n^{1/2})$

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- Open Question: Resolve the hardness of directed acyclic graphs

1. Min-Hop
2. Min-Fiber
3. Min-Both?

MinFiber

- *Input:* Supply network $G=(V,E)$, demand graph H, costs c_e to install a fiber in link e, and fiber capacity λ.
- *Solution:*

  a) Multiple l_e of fibers at link *e*

  b) Simple routing path P_h for each demand

  c) Assignment of one of λ colours to each P_h such that the number of paths of the same colour using any edge is at most l_e.
- *Goal:* Minimize $\Sigma\, c_e\, l_e$

Integer Decomposition Property

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- A polyhedron P has the integer decomposition property (IDP) if for any $x \in P$ and integer k such that kx is integral then we have

$$kx = \sum_{i \in [k]} x_i$$

where x_i is an integral vector in P.

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- Thm (Baum, Trotter): Matrix A is totally unimodular iff $\{x : Ax \leq b, x \geq 0\}$ has the IDP for every integer vector b.

WDM Flows on Directed Trees

- Thm: Exact solution MinFib on directed tree instances.
- Proof (Sketch):

  Let B be the matrix with $B_{ah}=1$ if routing for demand h goes through arc a. B and $[B^{T} I]^{T}$ are totally unimodular.

  Let l an allocation of fibers that satisfies capacity requirements.

  Define $P_l = \{x : B \cdot x \leq l, 0 \leq x \leq 1\}$ and note P_l is IDP.

  By assumption (1/λ, 1/λ, ... ,1/λ) is in P_l and hence there exists a decomposition of demands into λ classes such that each class can be assigned the same colour.

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Open Question

- Incompatible Assumptions:

  MinHop assumes an existing infinite capacity fiber in each link.

  MinFiber assumes full wavelength selective switching (i.e. infinite-arm Roadms)

- How can we unify both problems?

  In MinHop, consider purchasing extra fibers in each link at some cost.

  If we have to hop, can't we get a wavelength conversion for free?

Summary

MinFiber:
Exact Solution for Directed Trees
3.55 Approximation for Single-Source
via “Fractional implies Integral” results

MinHop:

	2-arm Roadms		*3-arm Roadms*
	Algorithm	*Hardness*	*Algorithm*
Undirected	$O(\log n)$	$\Omega(\log^{1-\epsilon} n)$	$O(\log n)$
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DAG	$O(n^{1/2})$	$\Omega(\log n)$	$O(n^{1/2})$