Graph Distances in the Streaming Model

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& Jian Zhang

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- What about graph problems?

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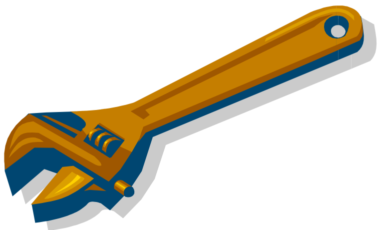
- We focus on the memory limit O(n polylog n).

Overview

- Efficient Spanner Construction
- Lower Bounds for Approximation Factors
- Hardness of Constructing BFS

1. Spanner Construction



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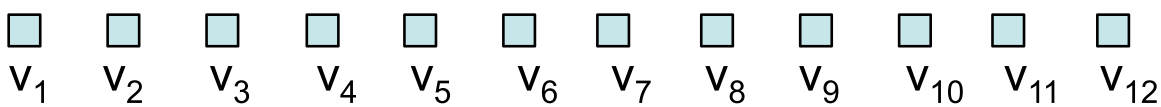
- Main result: Randomized algorithm for log *n*-spanner using *O(n polylog n)* space and *O(polylog n)* time per edge.
- Naive Algorithm: If an edge completes a short cycle, ignore it.

Our Algorithm for a 2*t*+1 Spanner

- Assign an initial set of labels L(*v*) to each vertex *v*:



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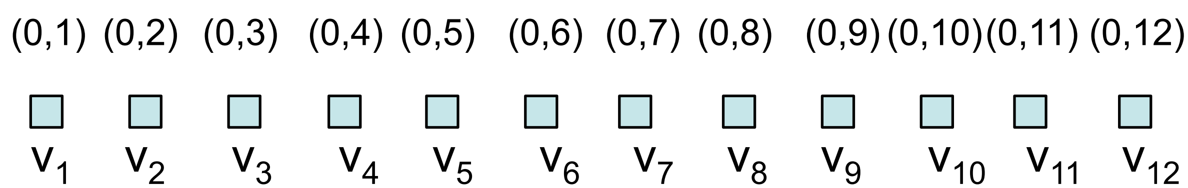

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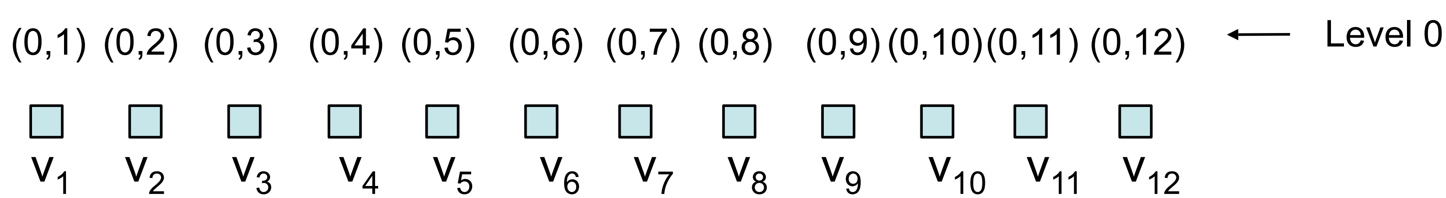

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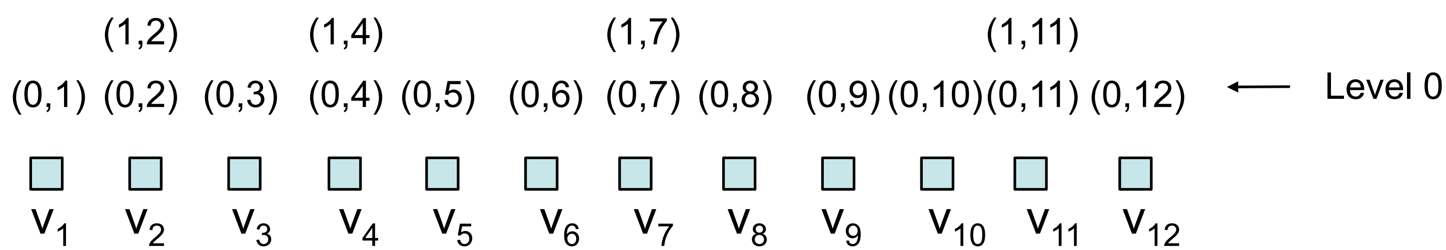

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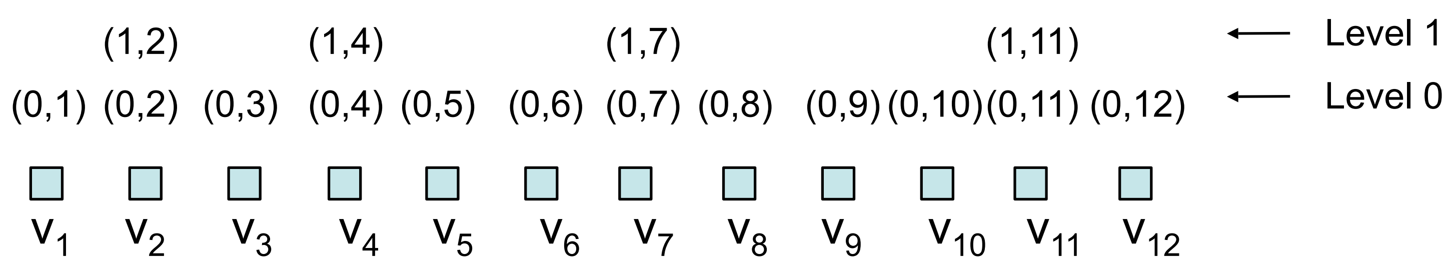

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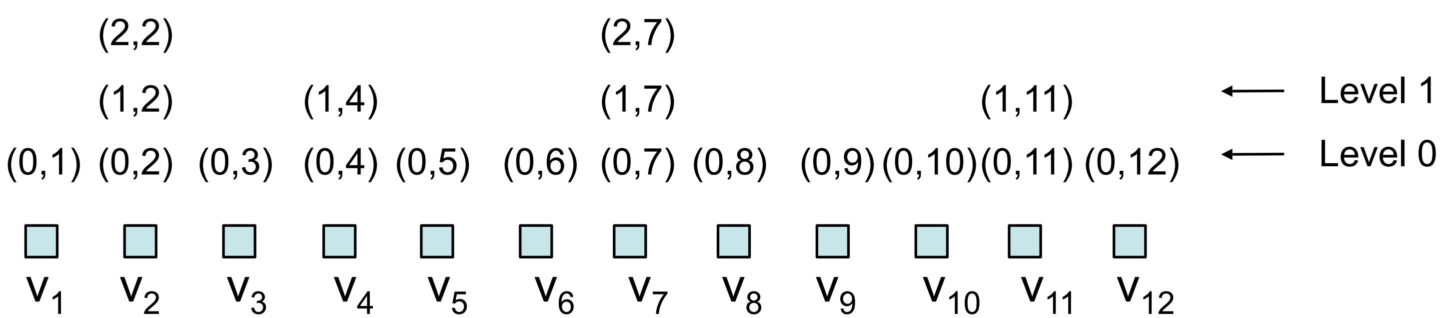

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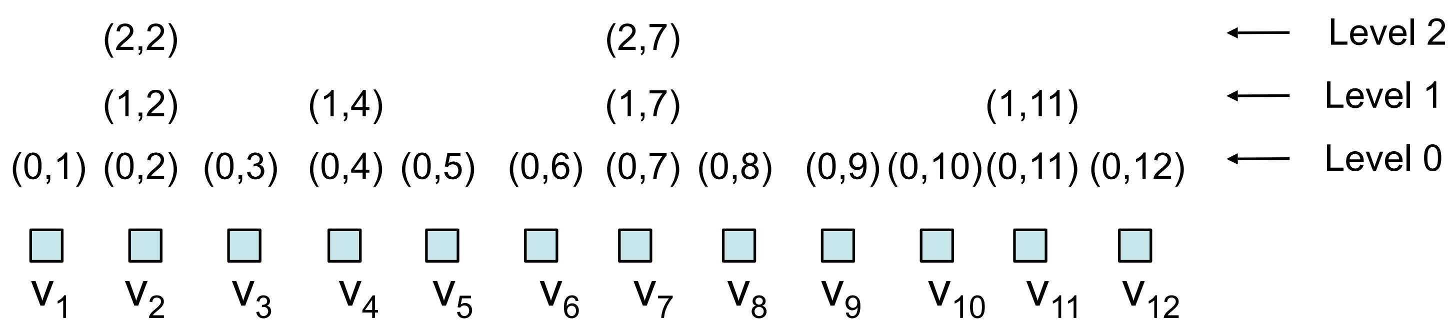

Our Algorithm for a $2t+1$ Spanner

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	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
Level 2		(2,2)					(2,7)					
Level 1		(1,2)		(1,4)			(1,7)				(1,11)	
Level 0	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)	(0,8)	(0,9)	(0,10)	(0,11)	(0,12)

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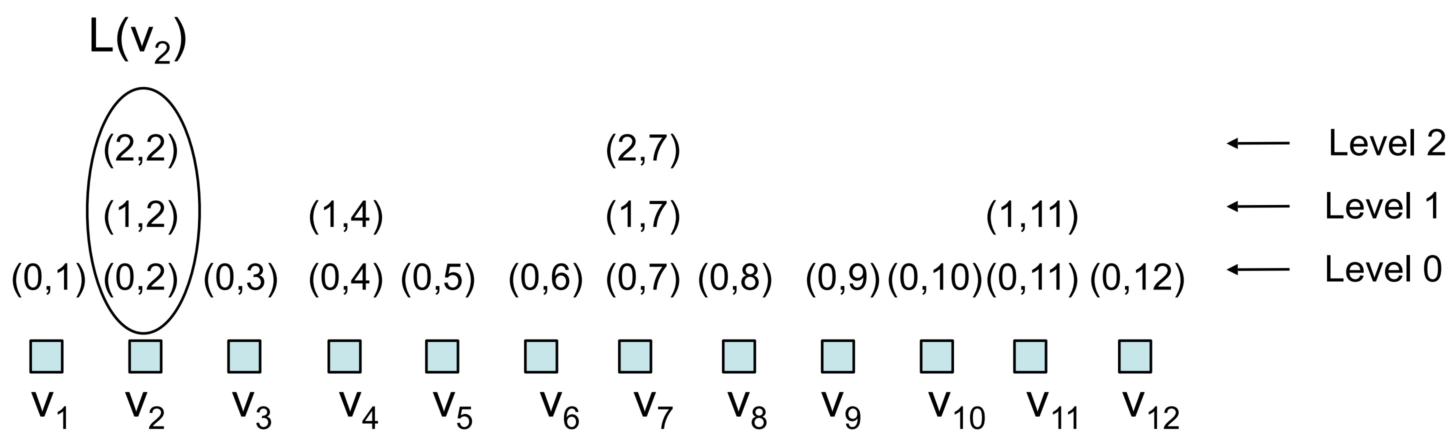

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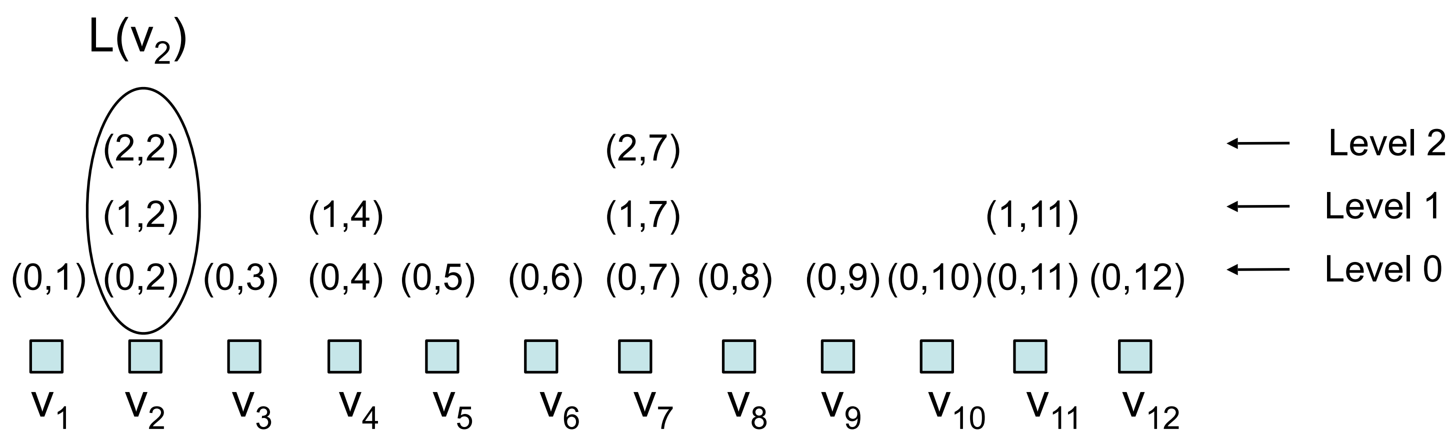

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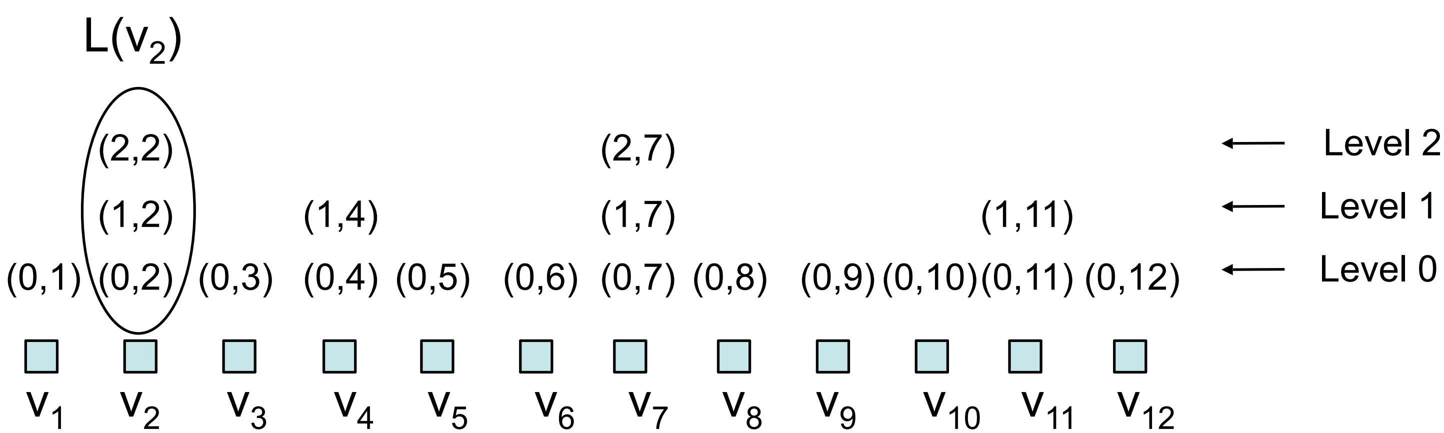

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- As edges stream in we grow L(.) and C(.)

Types of edges in Spanner:

- Spanning trees for each C(*l*)
- Set T of edges connecting clusters at top level
- For each vertex v, set of edges M(*v*) to neighbours of *v*.

Invariants:

- Each vertex occurs in at most one cluster per level.
- If $l \in L(v)$ and $l' = Succ(l)$ then $l' \in L(v)$

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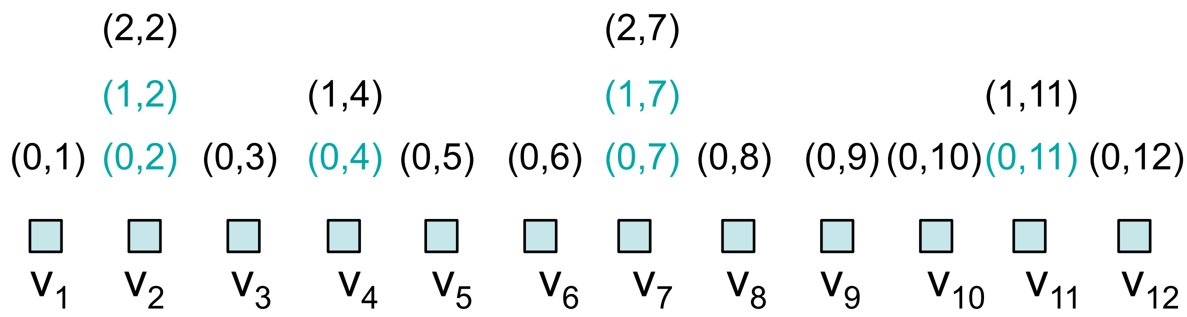
- Let $L_u(v)$ = all labels in L(*u*) higher than L(*v*):
  - If $L_u(v)$ contains a selected label *l*, choose the smallest and add all the successors (*l'*, *l"…*) of *l* to *L(v).* Add *e* to C(*l'), C*(*l"*)...
  - If $L_u(v)$ doesn't contains a selected label, add *e* to M(*v*) if it doesn't contain a *u'* with the same label as *u.*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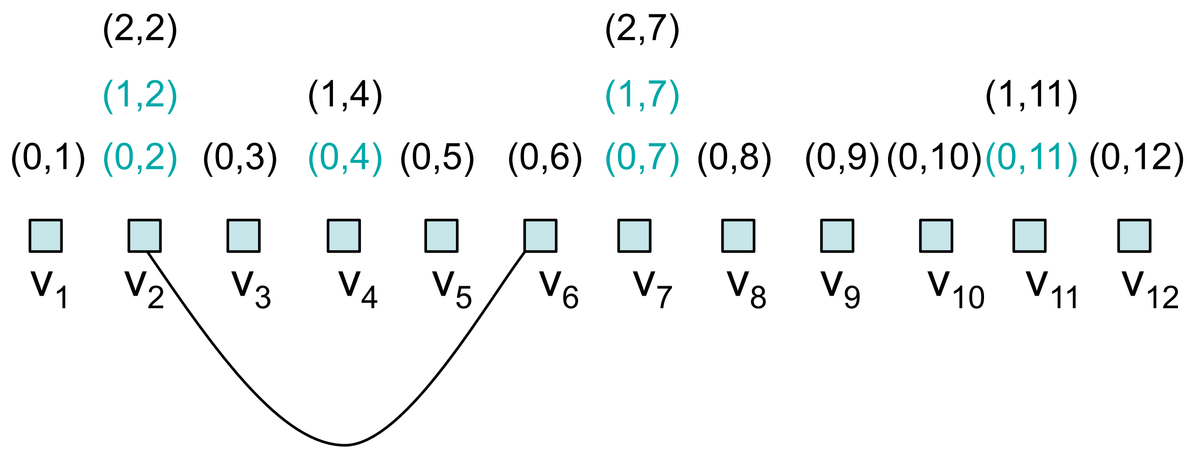

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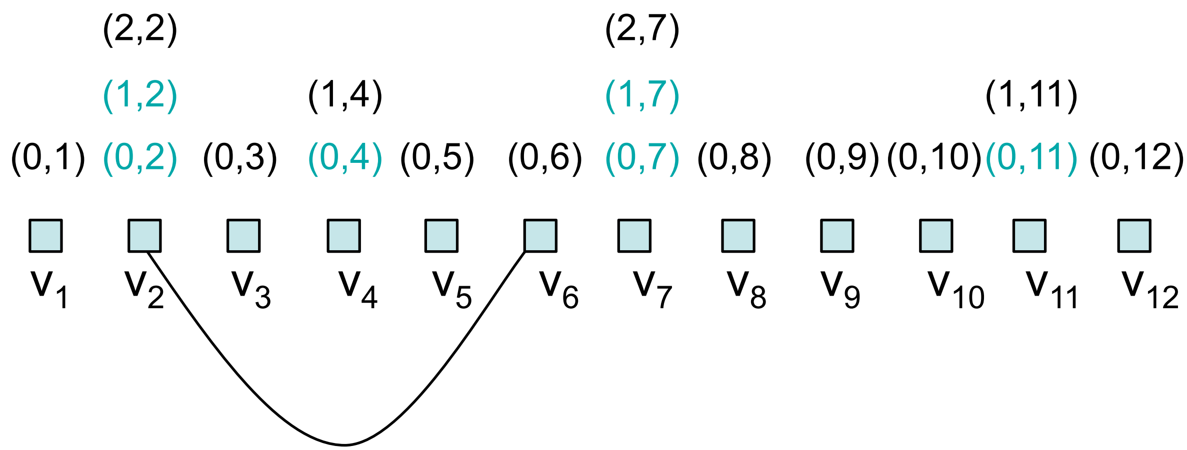

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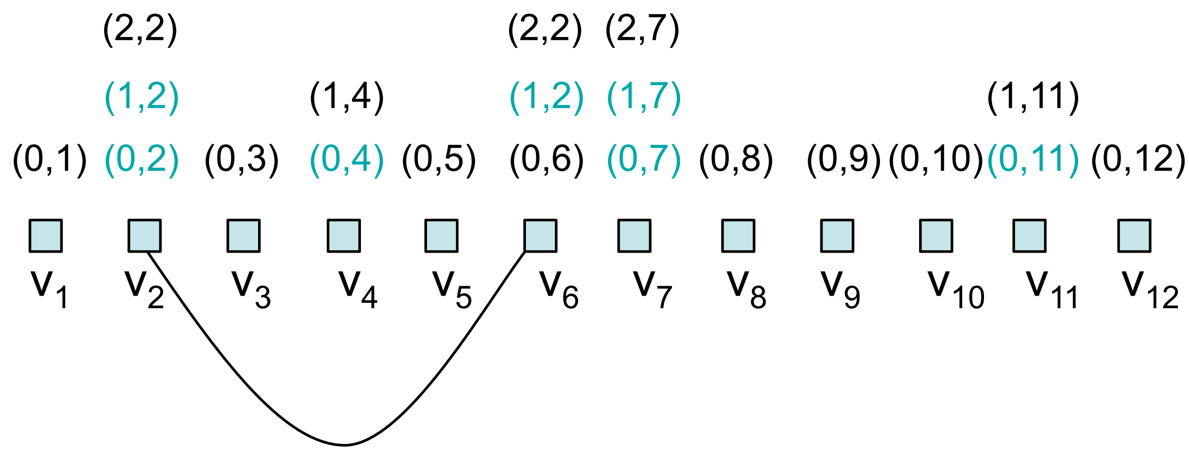
Add (v_2,v_6) to spanning trees for clusters C(1,2) and C(2,2).



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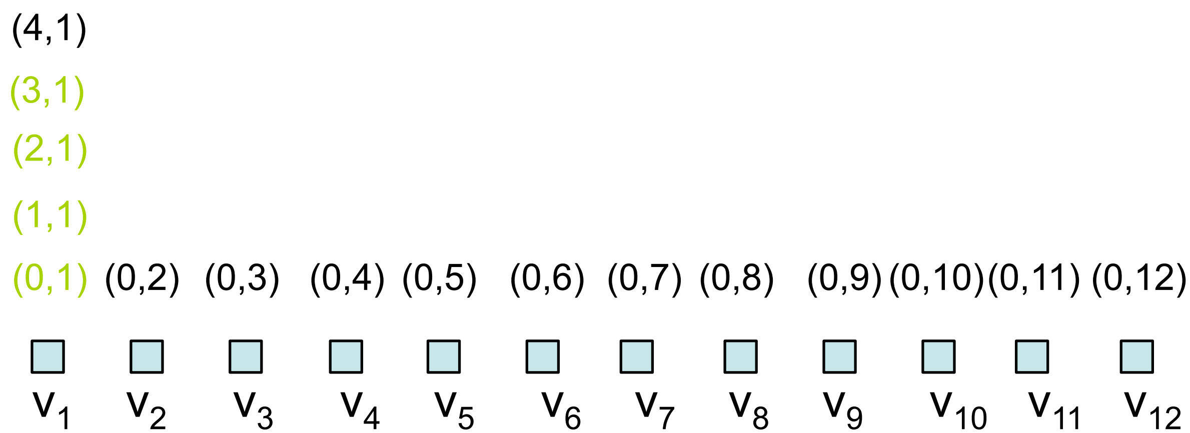
- Spanning tree of i^{th} level cluster has diameter $\leq 2i$



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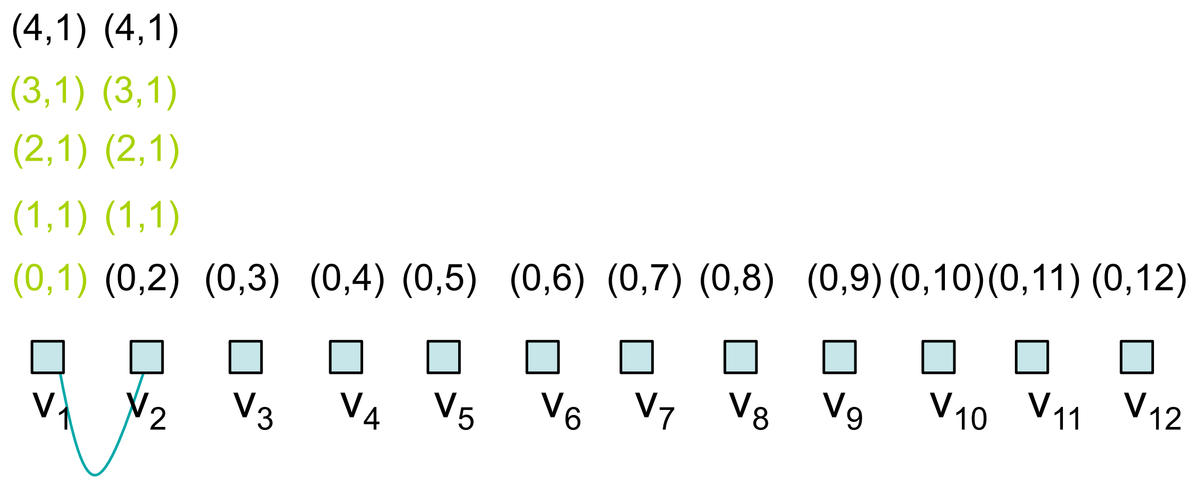

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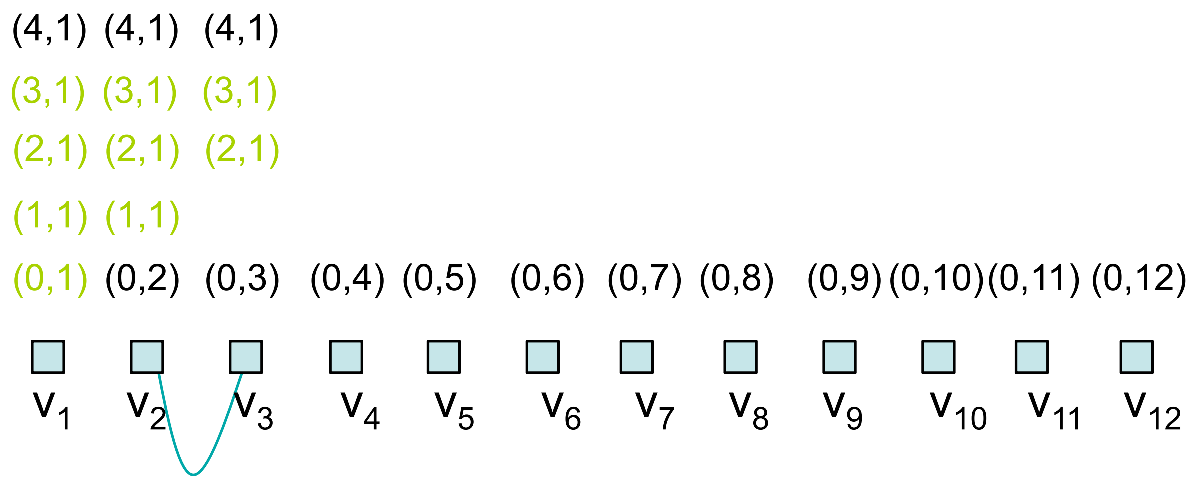

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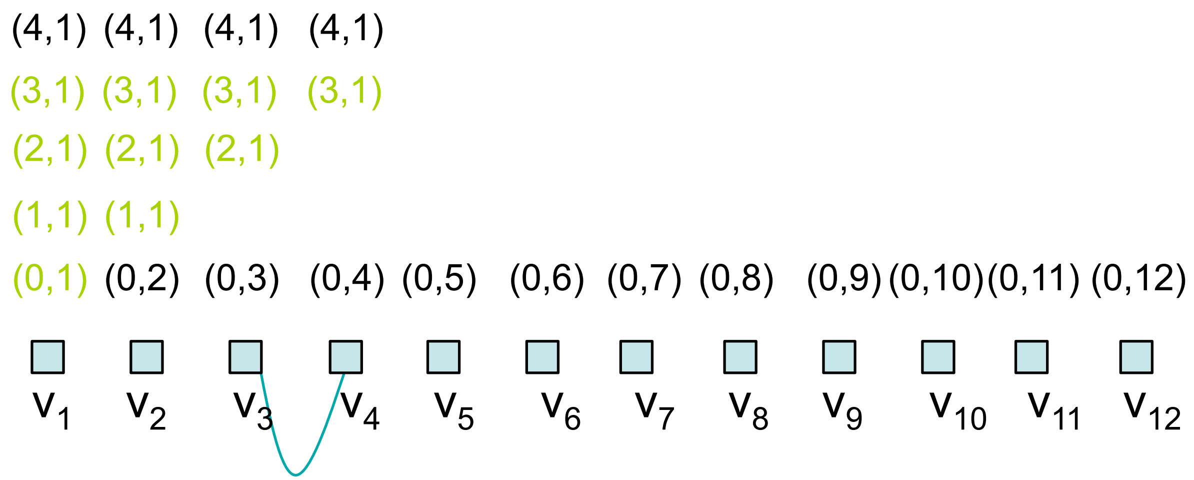

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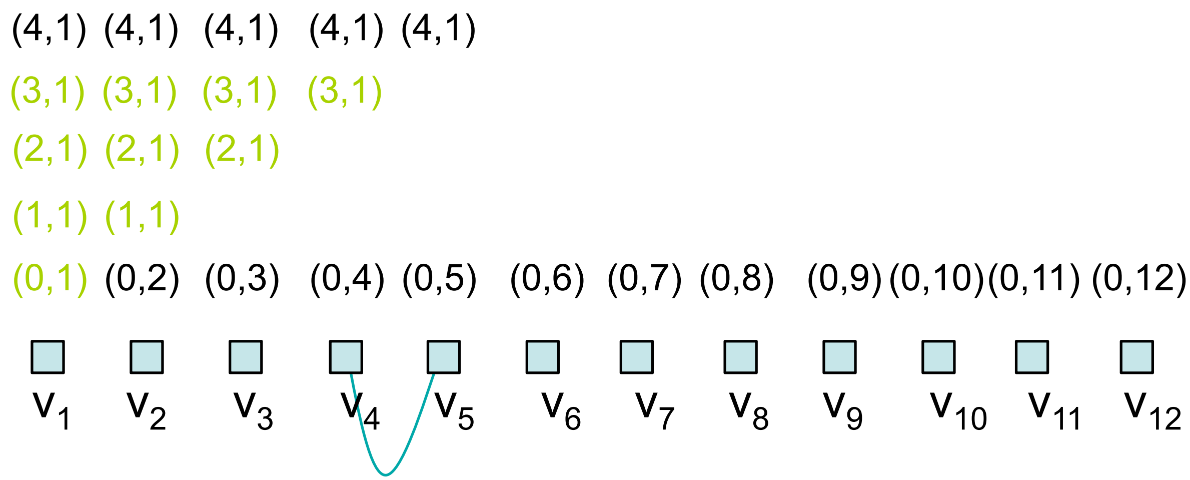

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- Spanning tree of i^{th} level cluster has diameter ≤ 2*i*
- For each edge ignored there is a short detour:
  - If *u* and *v* have an label *l* in common they are ≤ *t* apart.
  - If *u* and *v* have top level labels then they are ≤ 2t+1 apart.
  - If (*u*,*v*) not added to M(v) there was an edge (*u'*,*v*) in M (*v*) with *u'* in same cluster as *u*. Then they are ≤ *t+1* apart.

2. Lower Bounds

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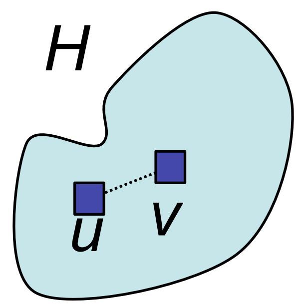

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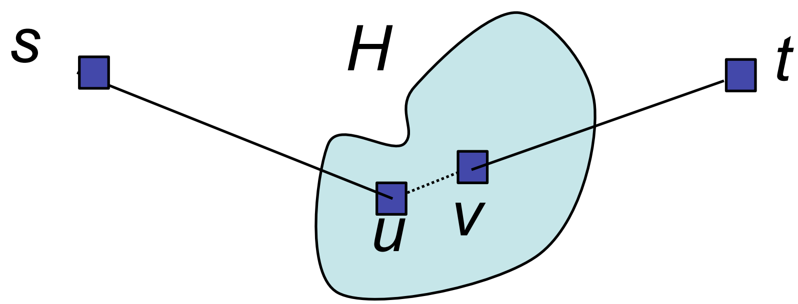

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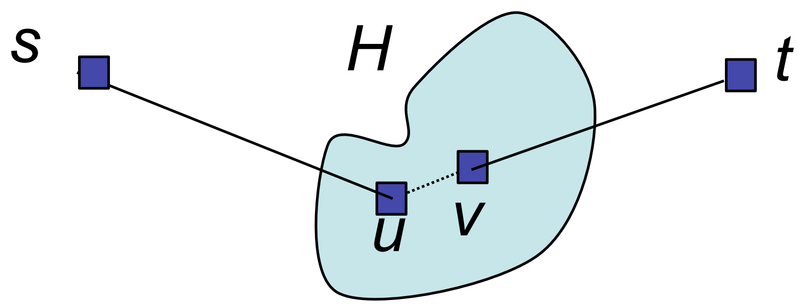
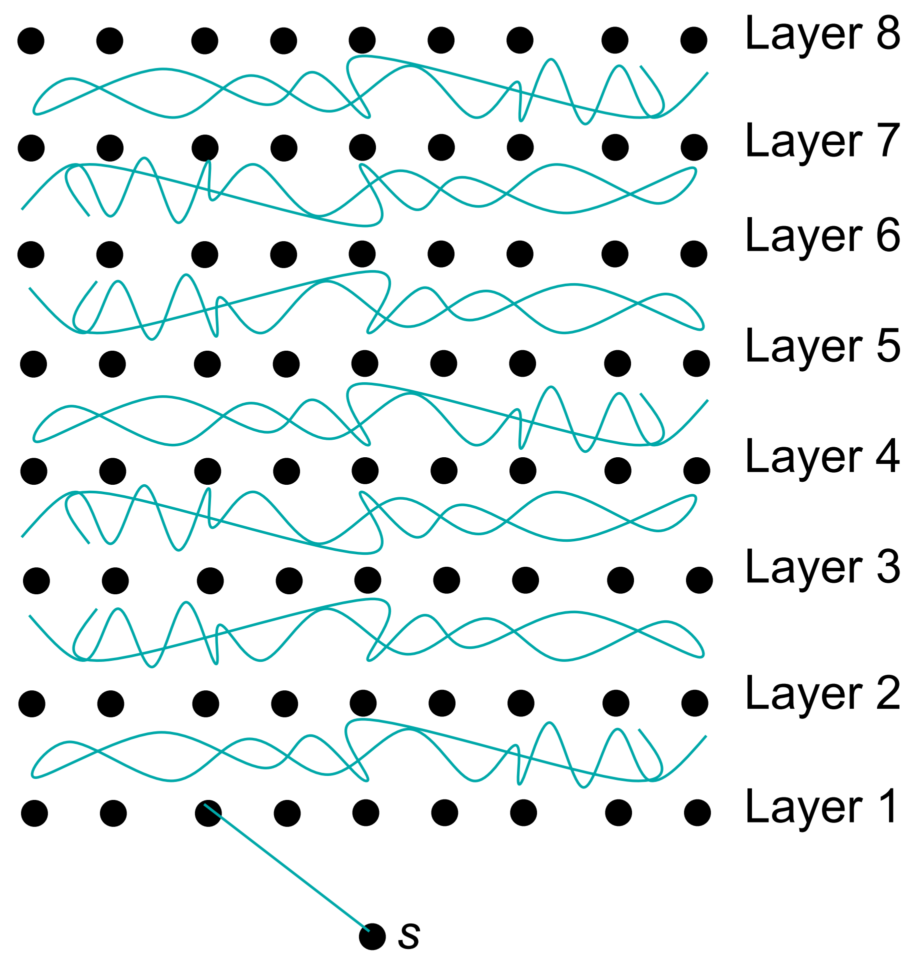


- In one pass it is not possible to approximate the distance to better than $1/\gamma$ factor using space $o(n^{1+\gamma})$.

3. Hardness of Computing a BFS

Layered Graph



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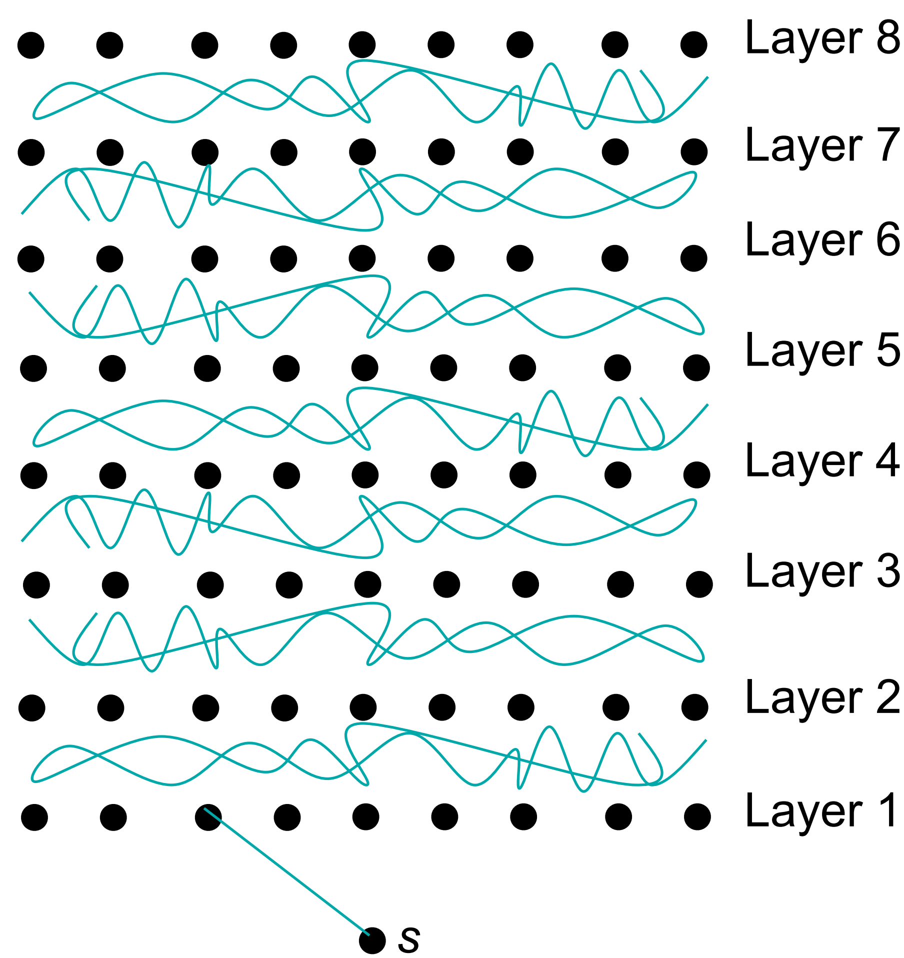
Problem:
Find all vertices in layer *i* that are a distance *i* from vertex *s*.



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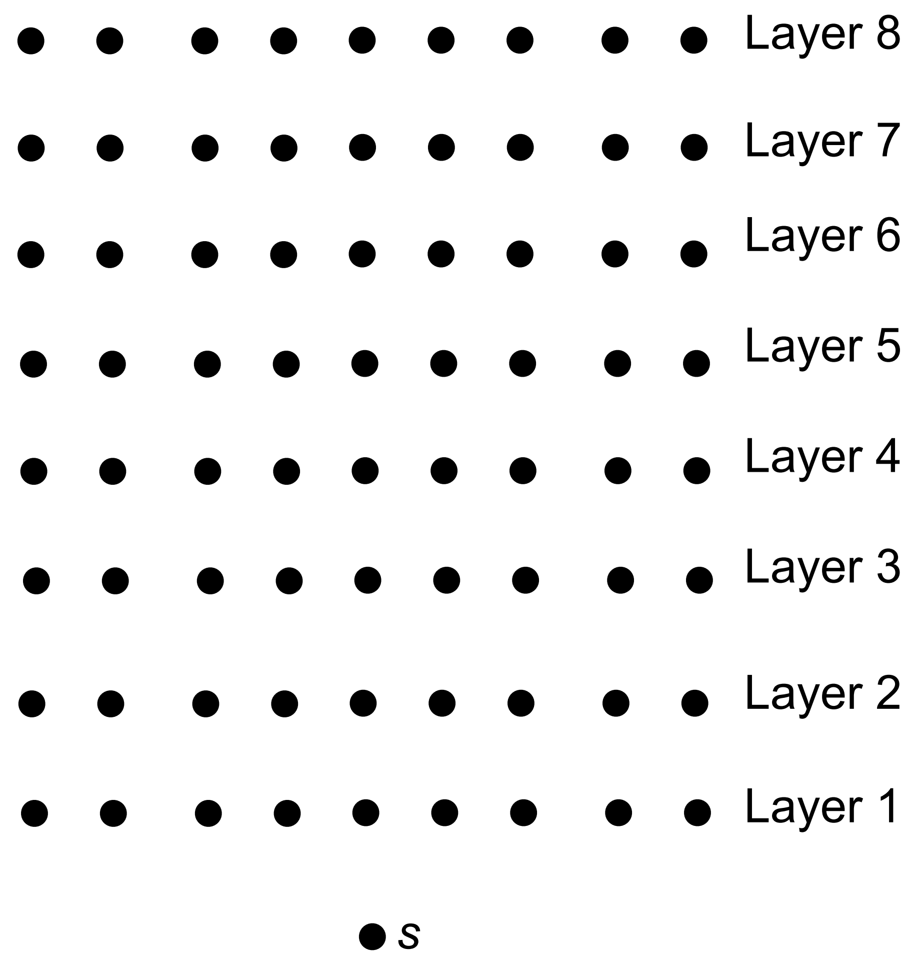

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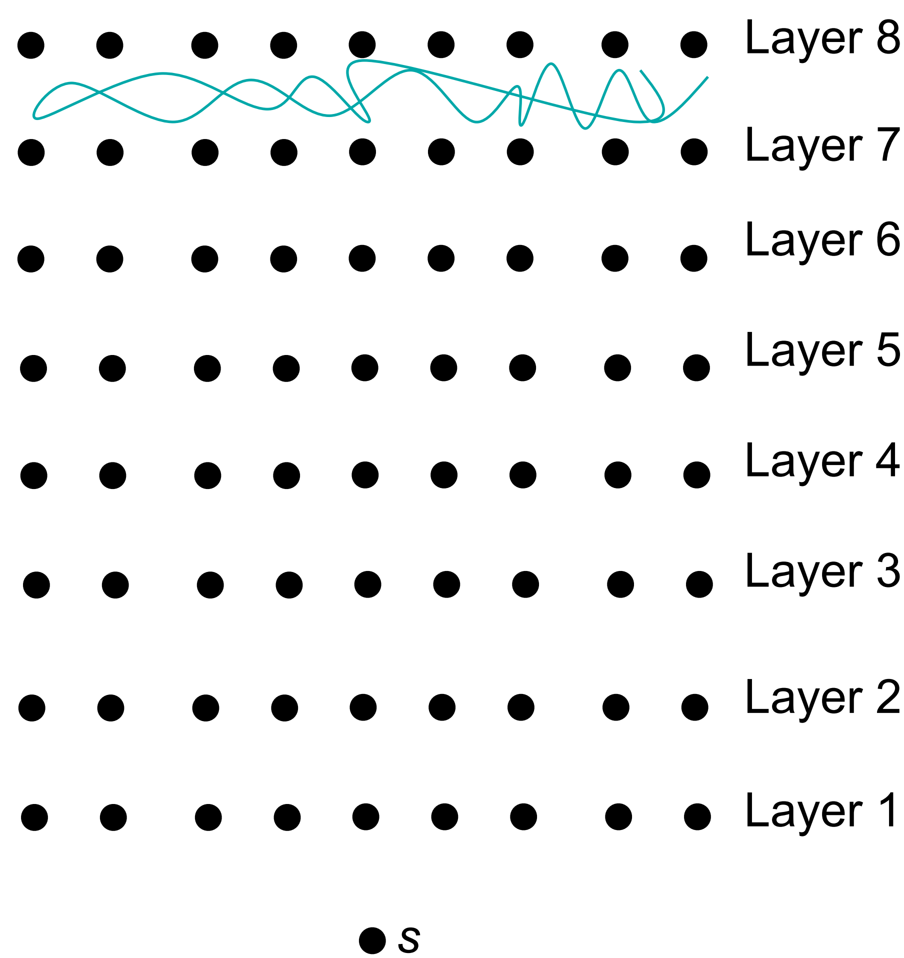

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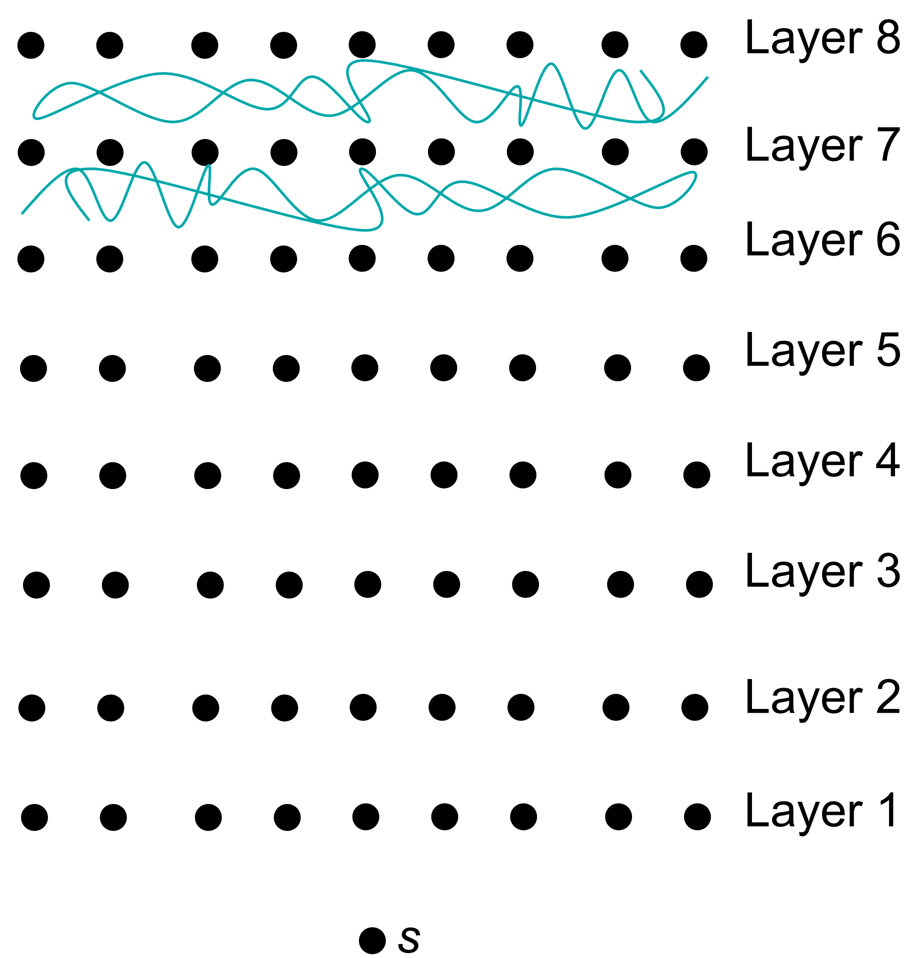

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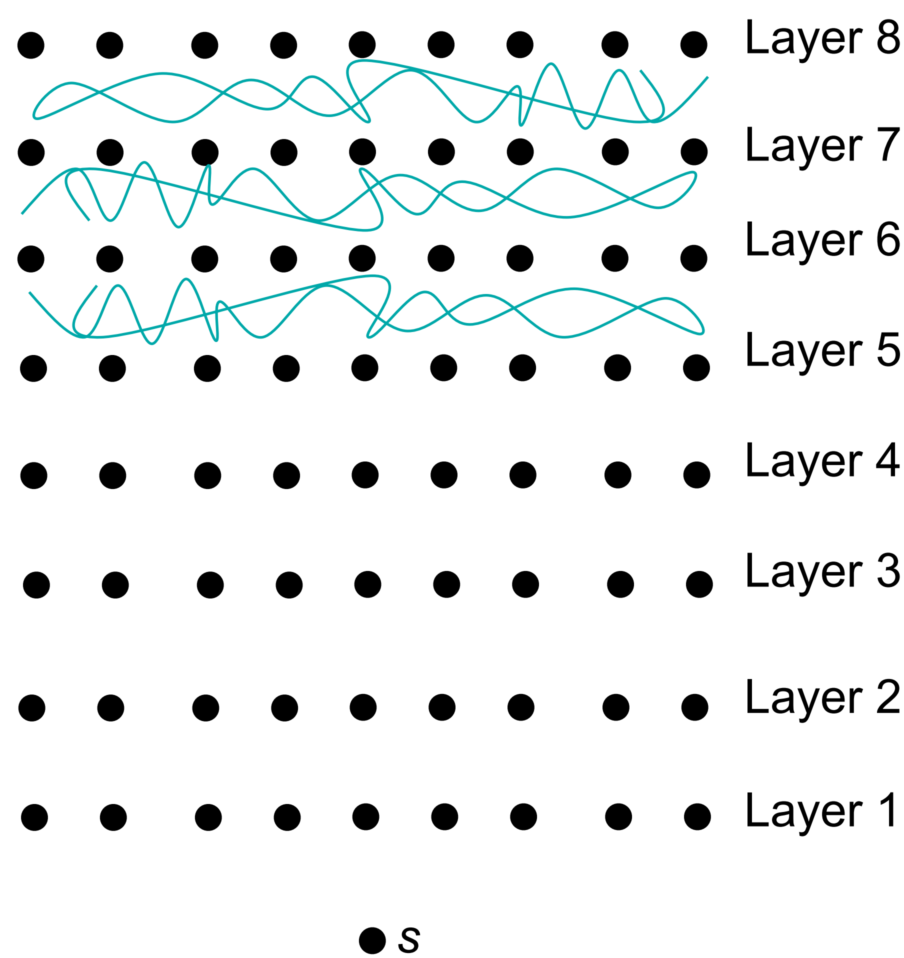

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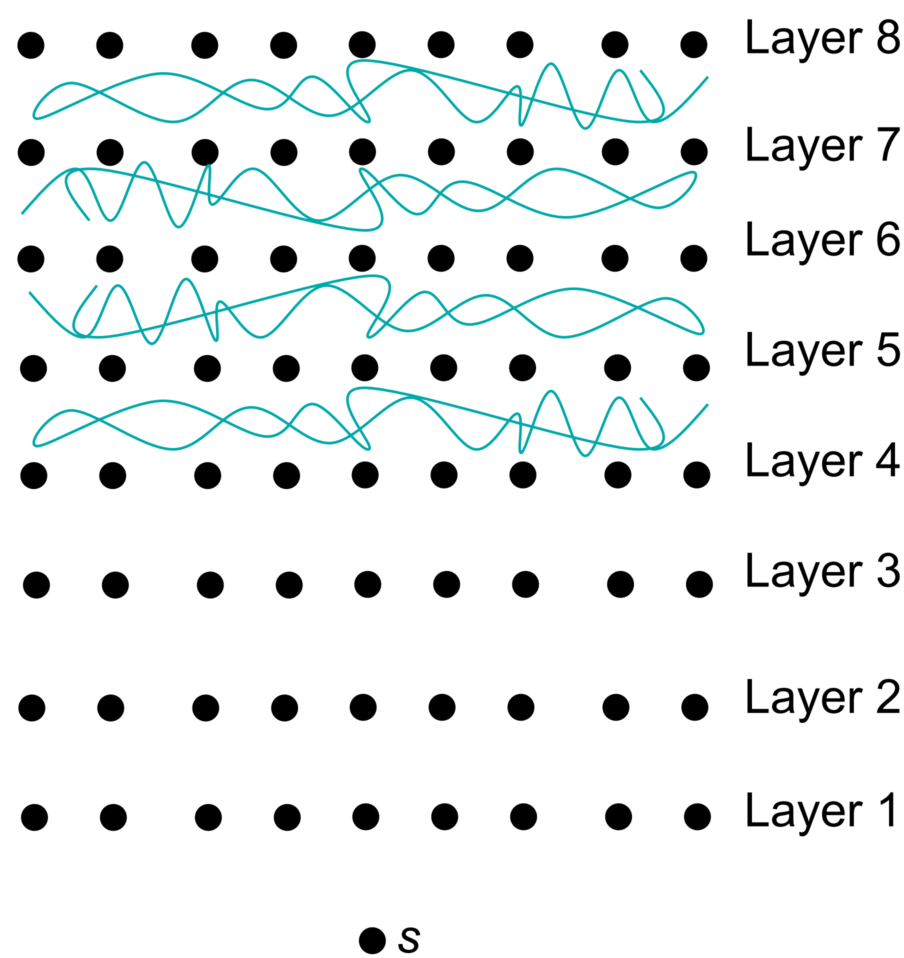

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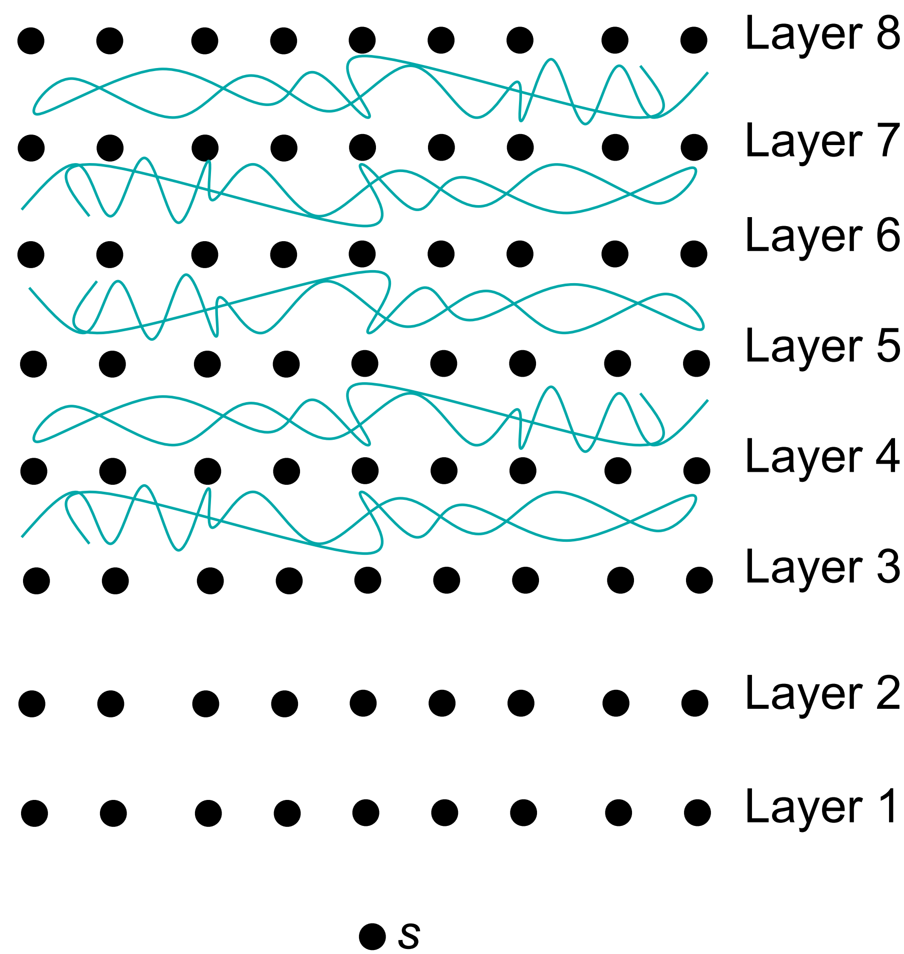

Layered Graph

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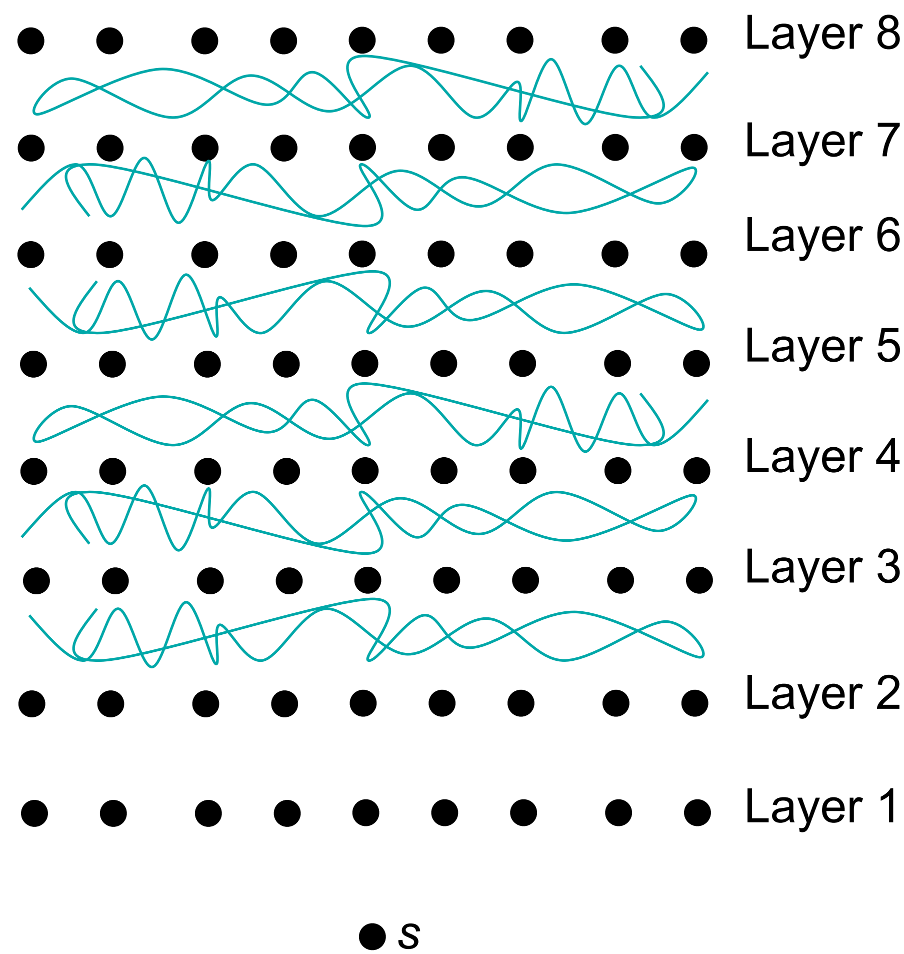

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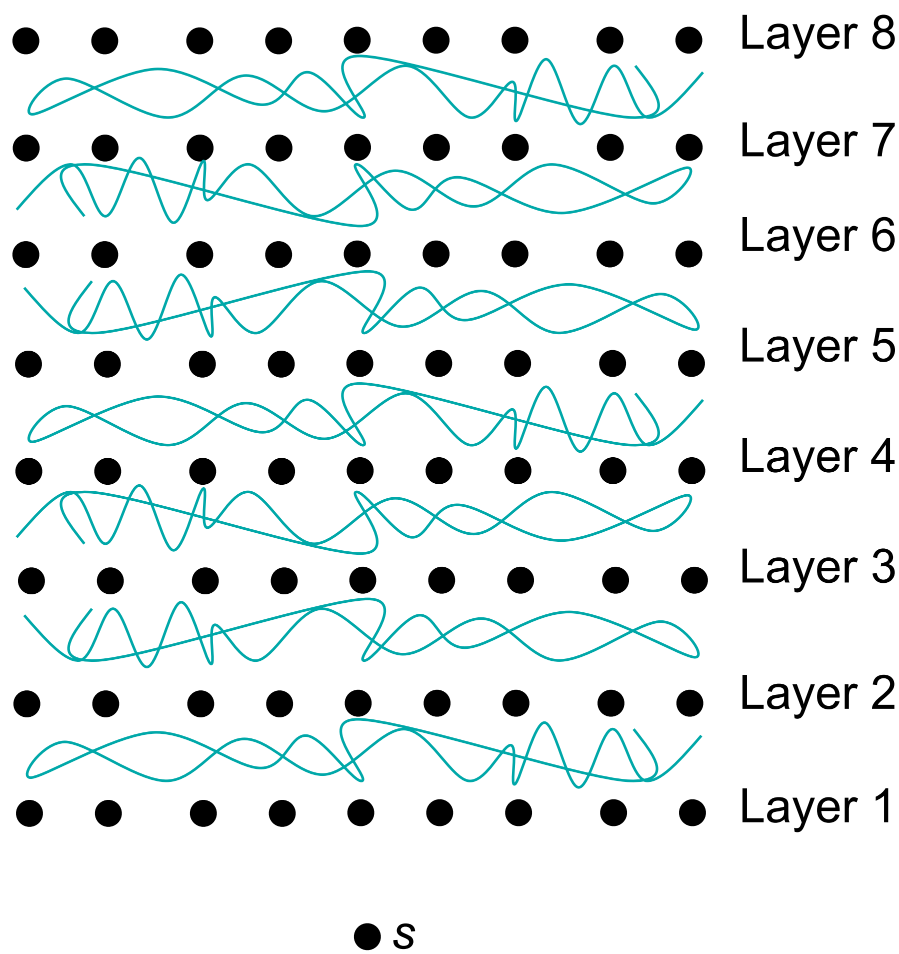

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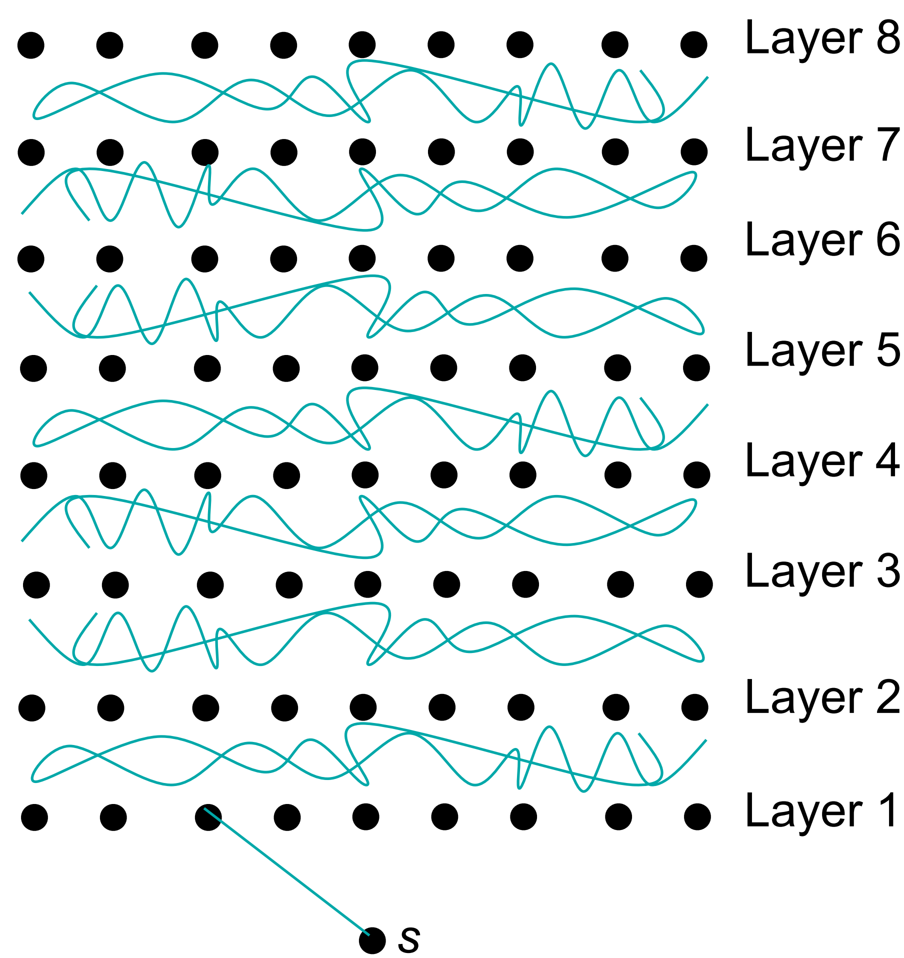

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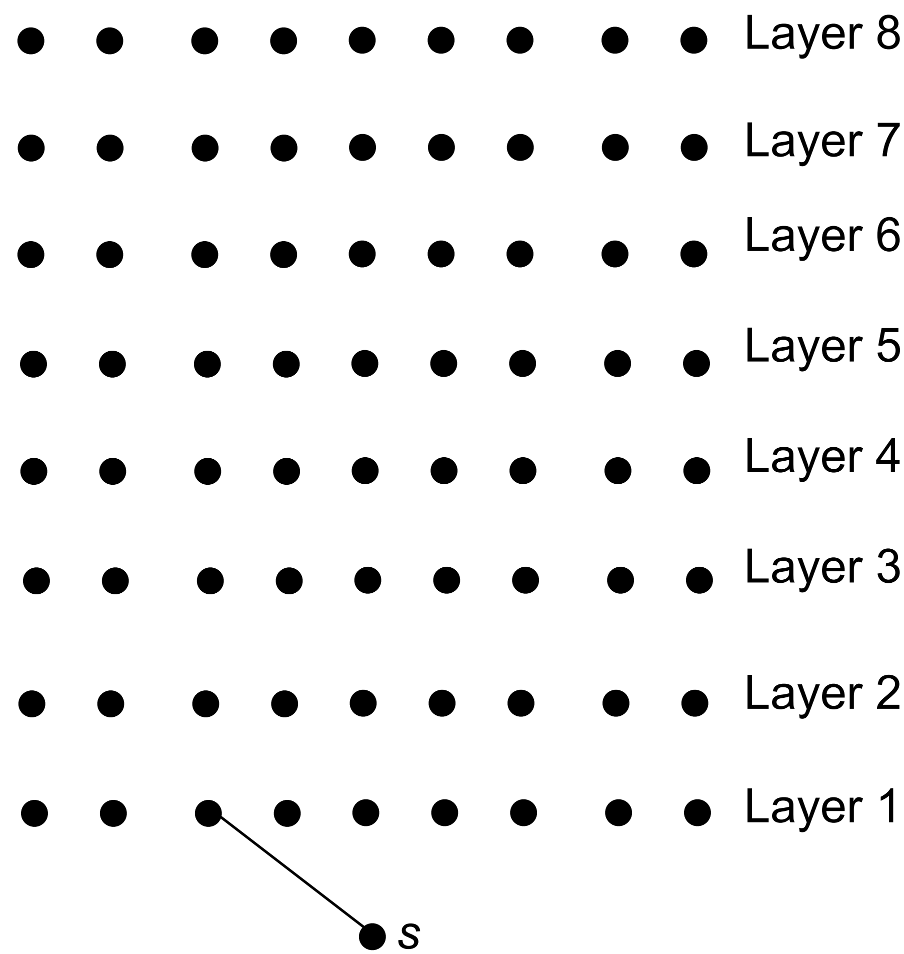

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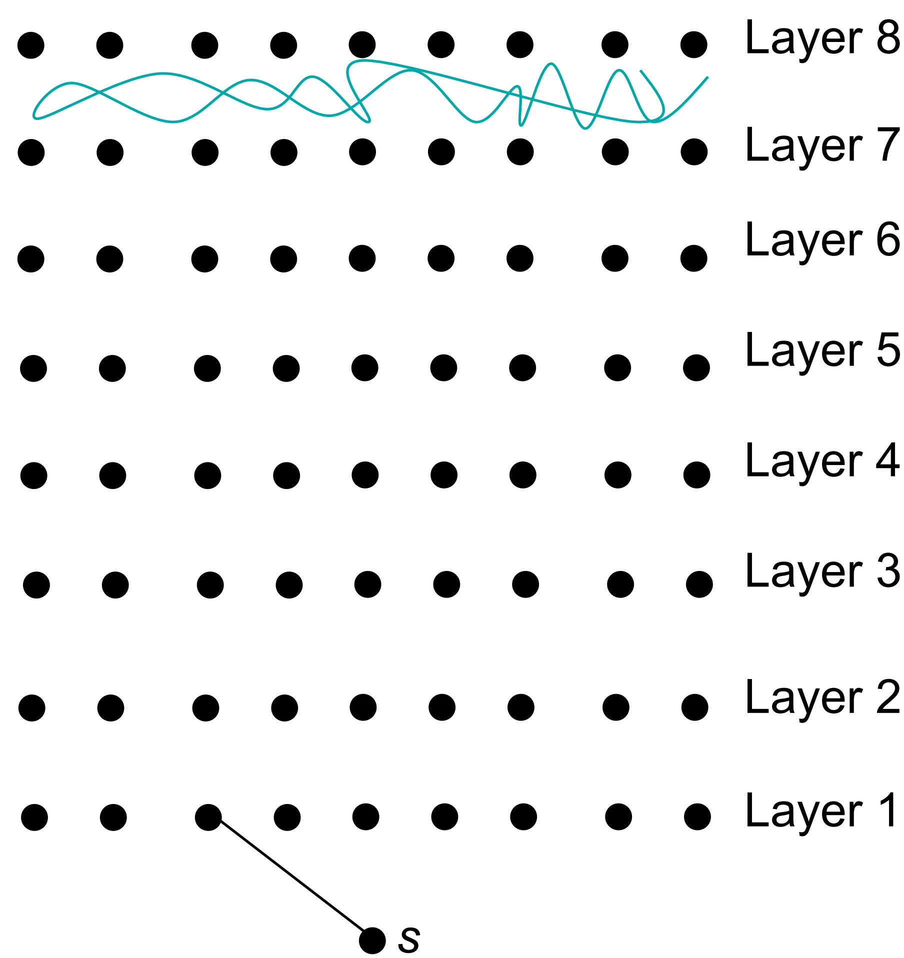

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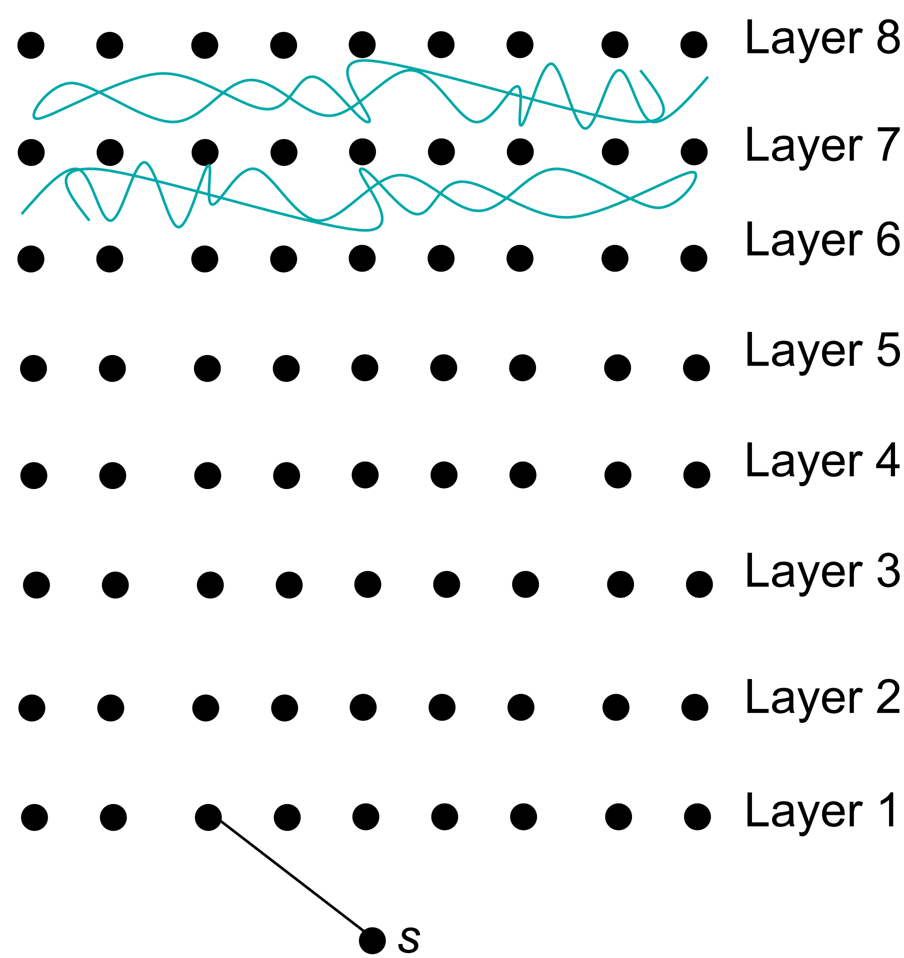

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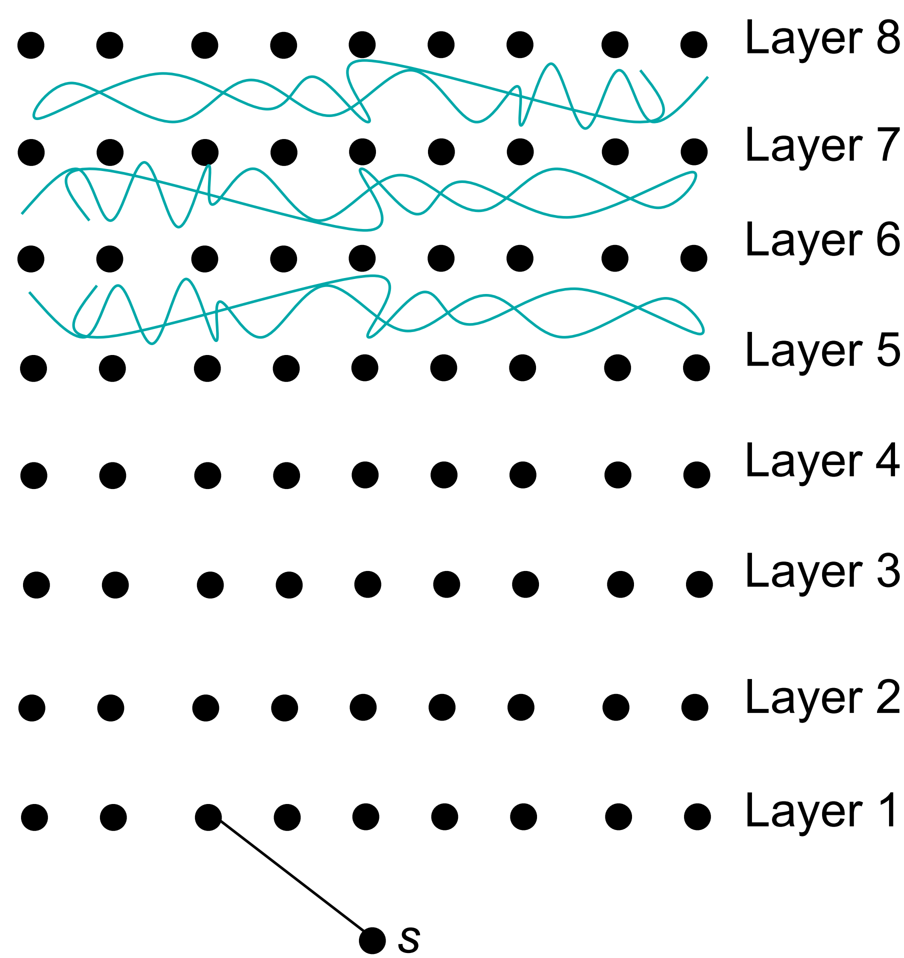

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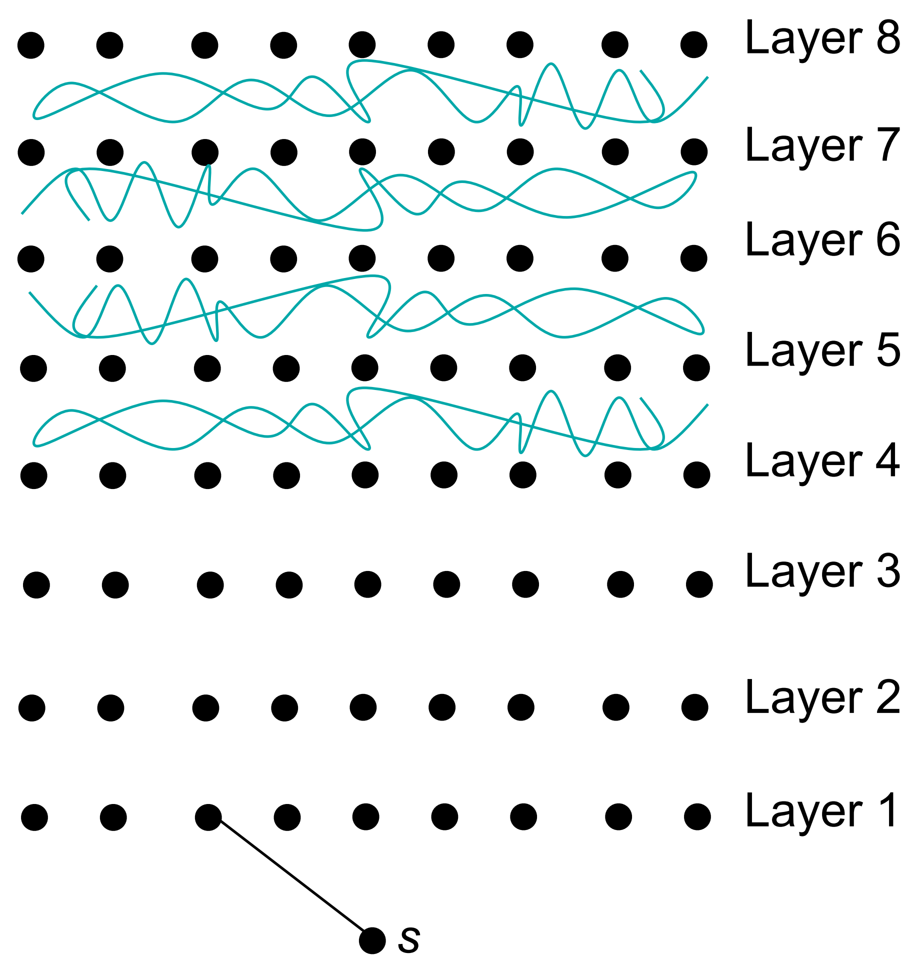

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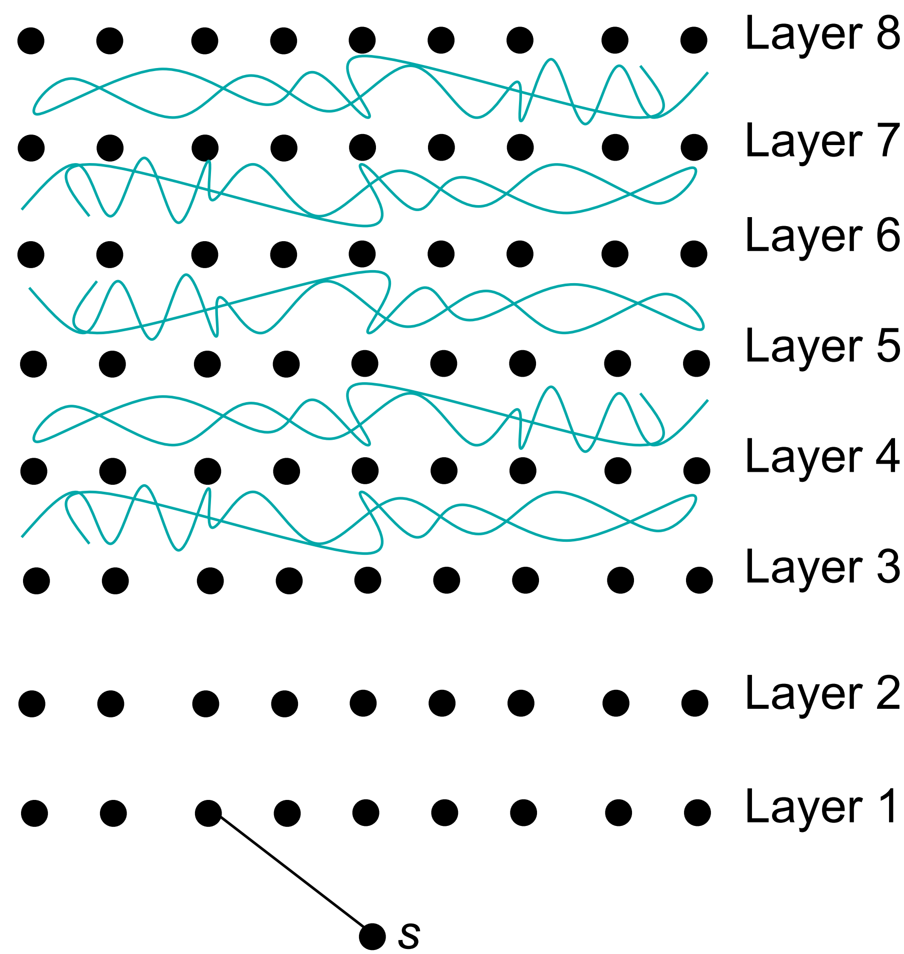

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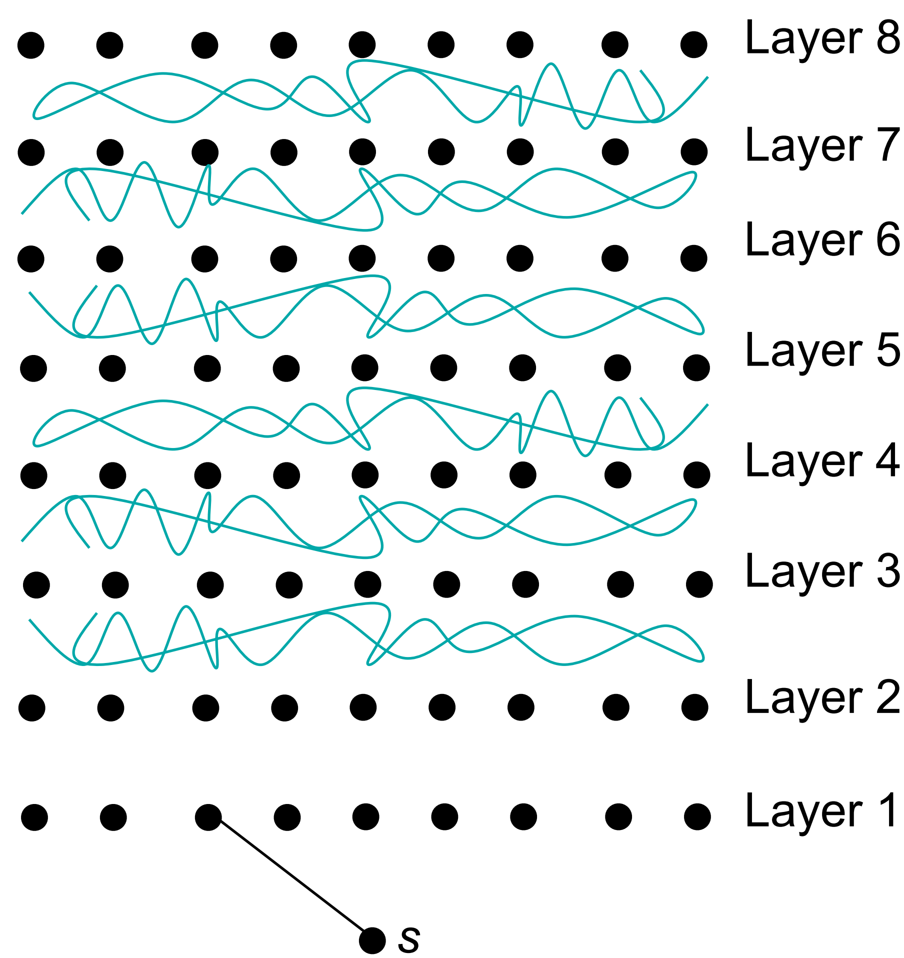

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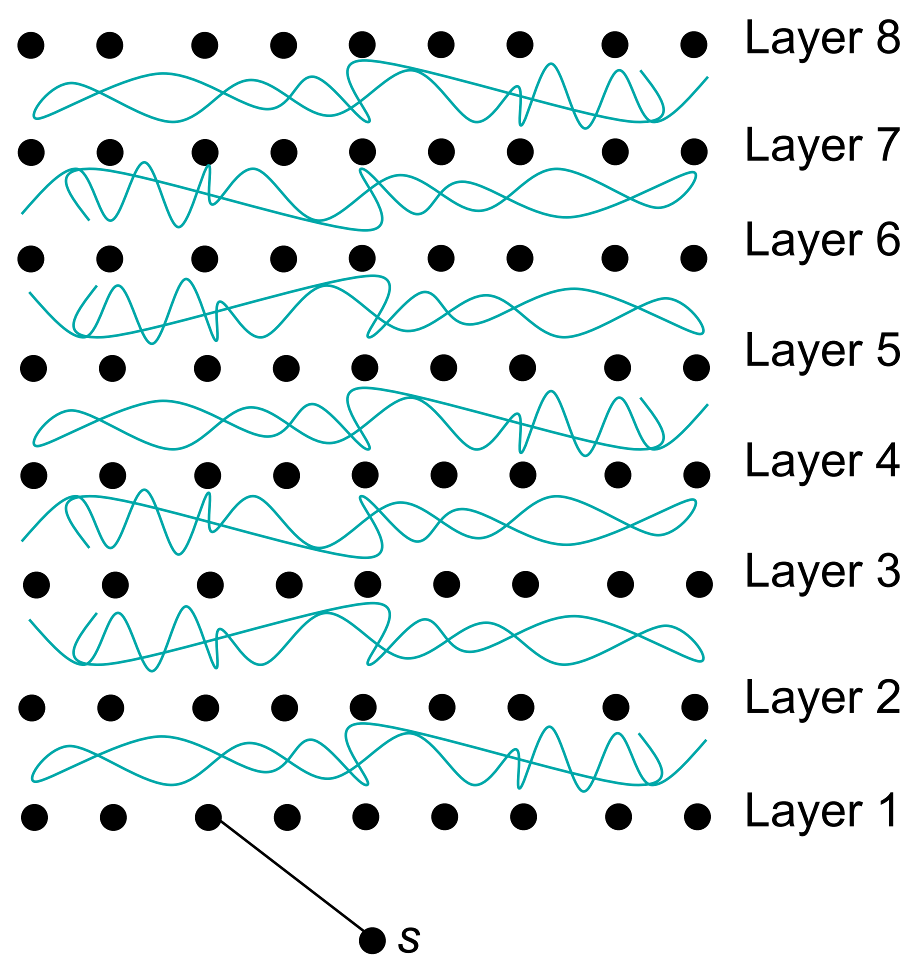

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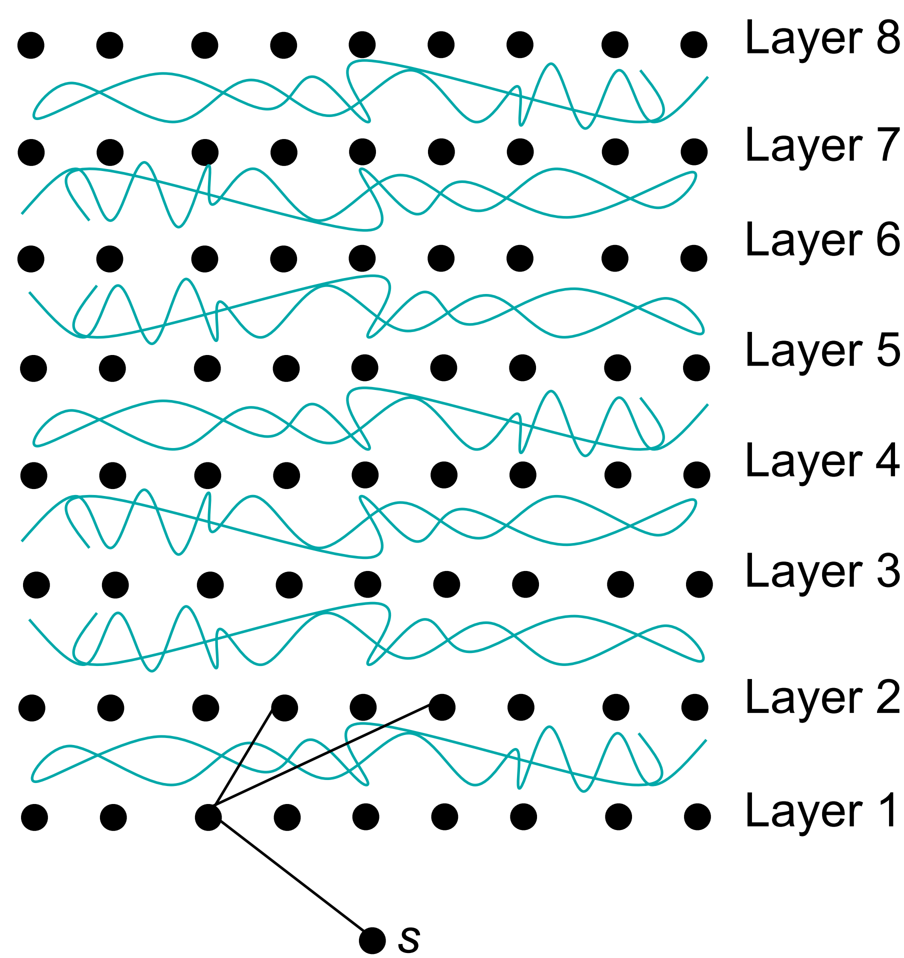

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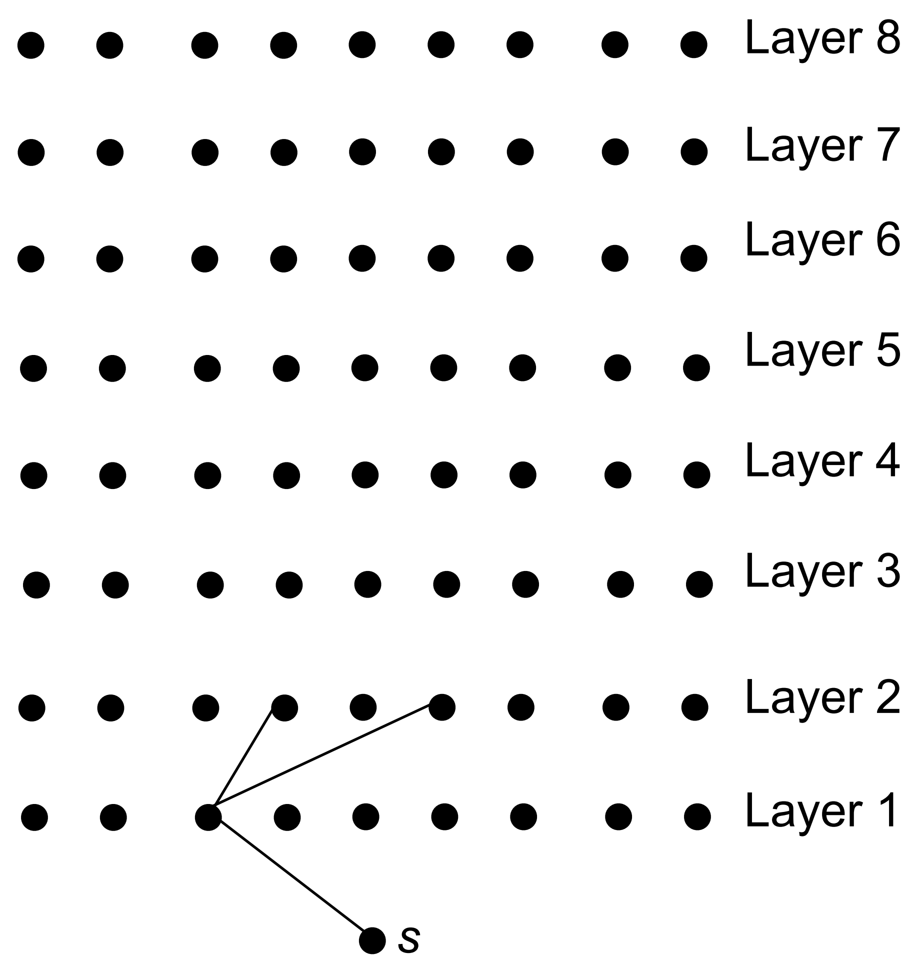

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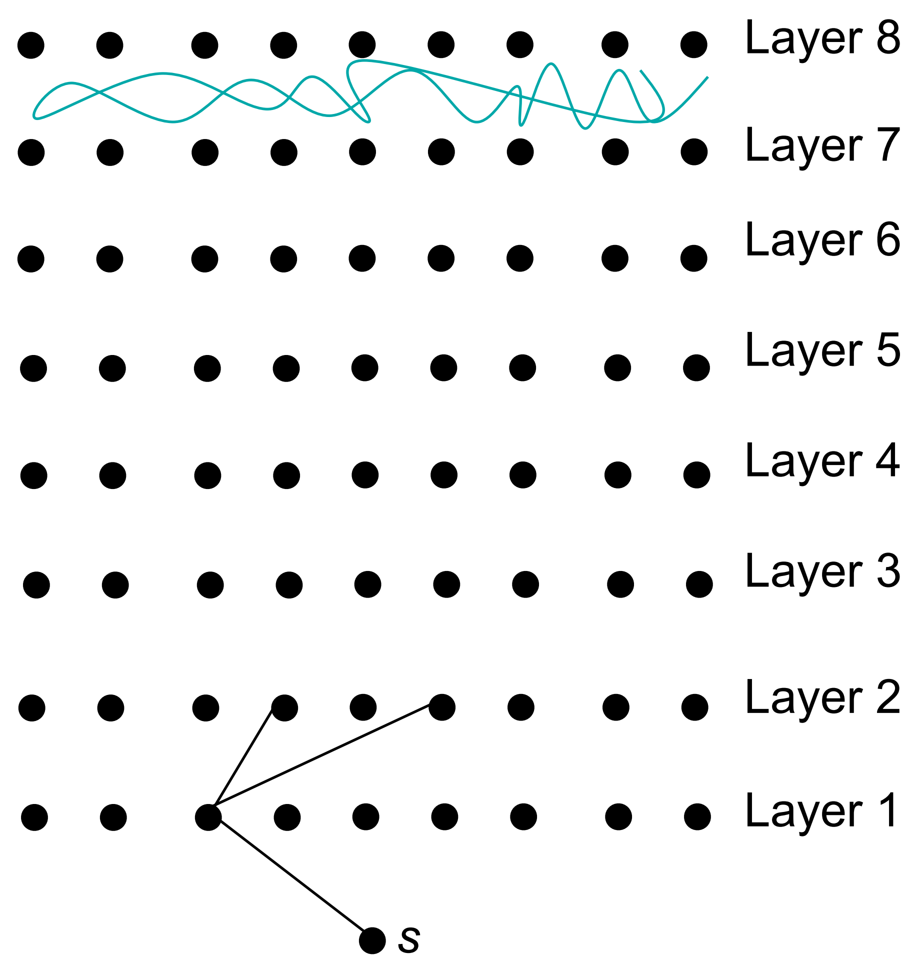

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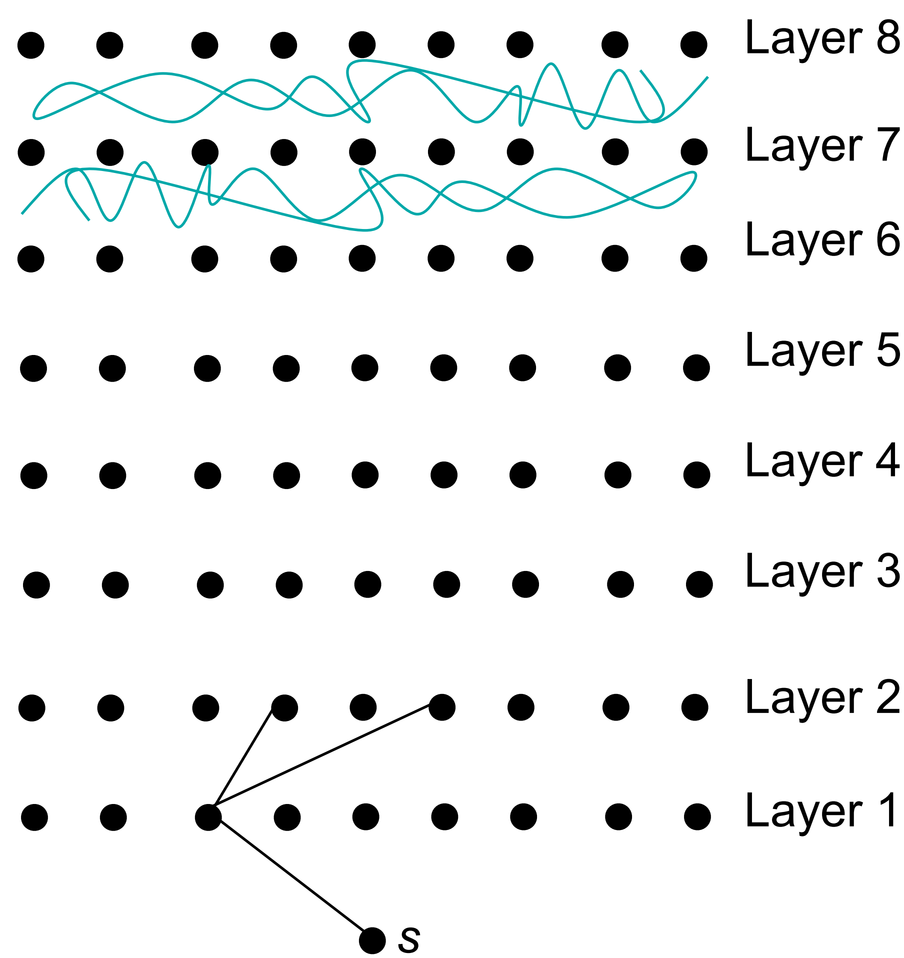

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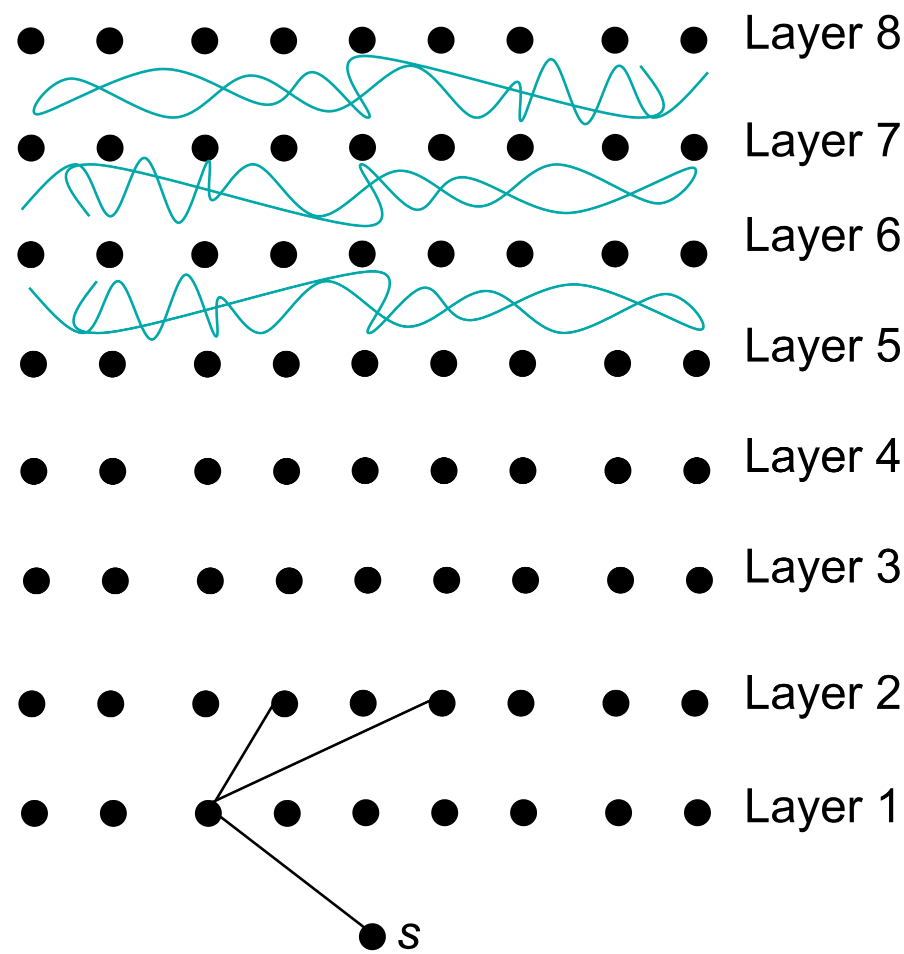

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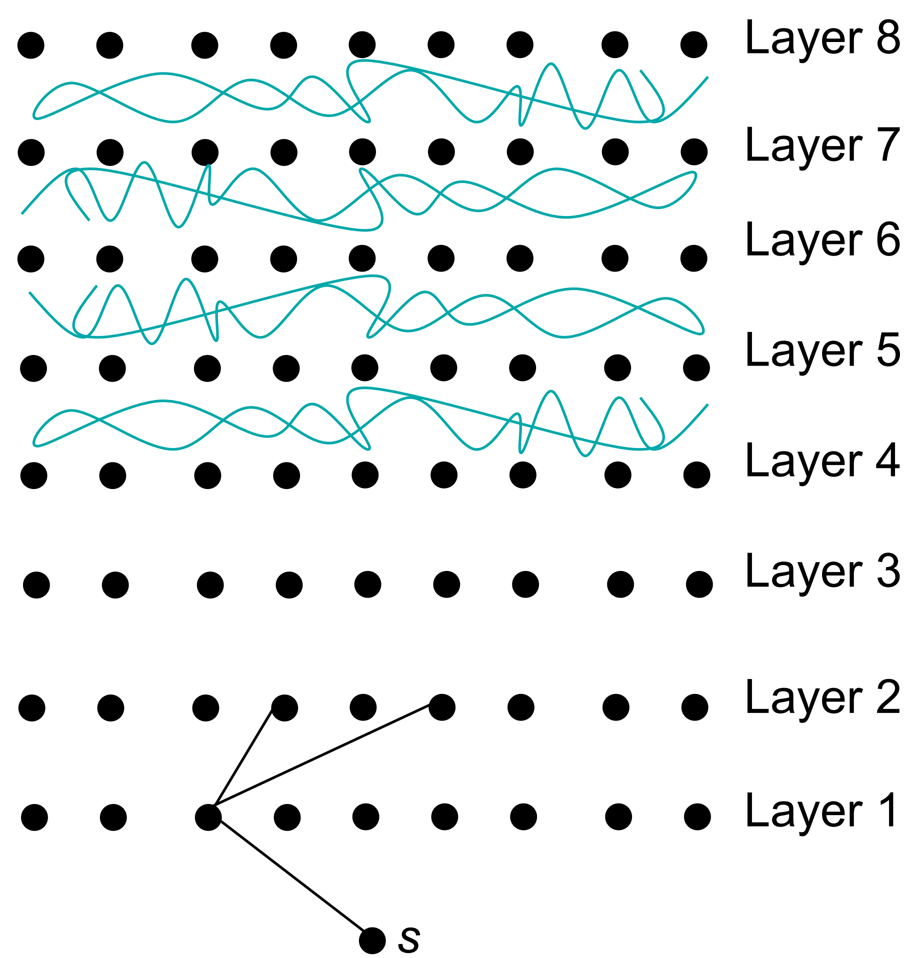

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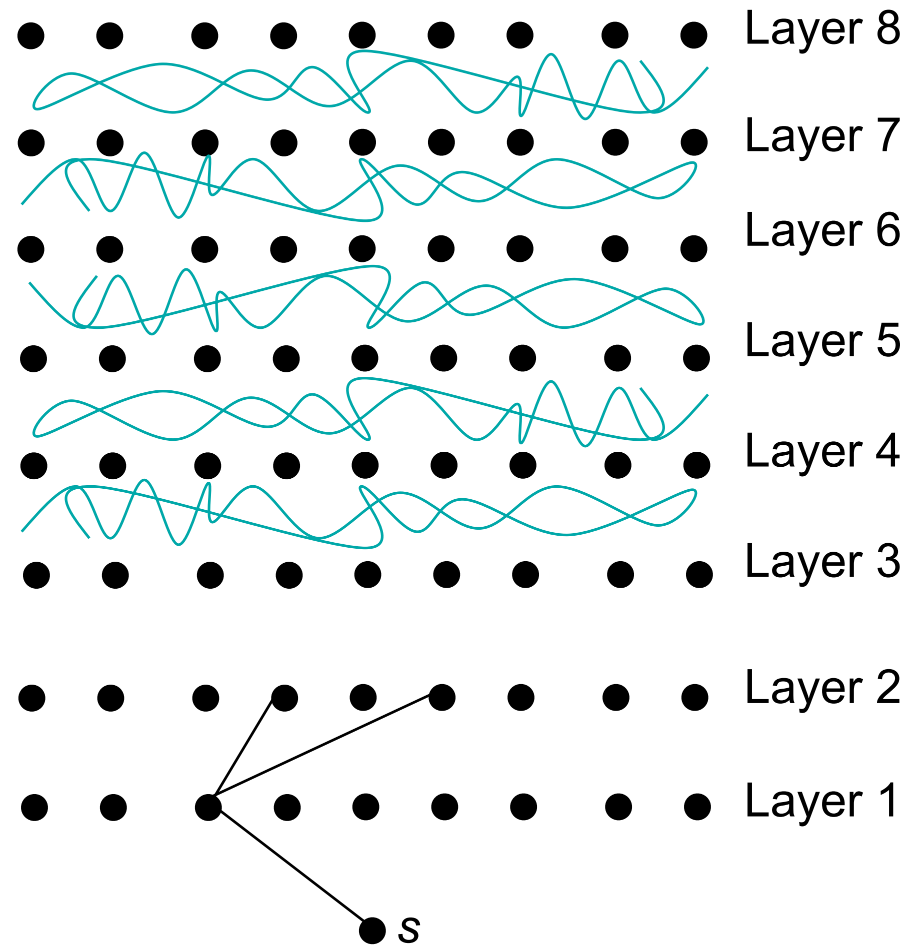

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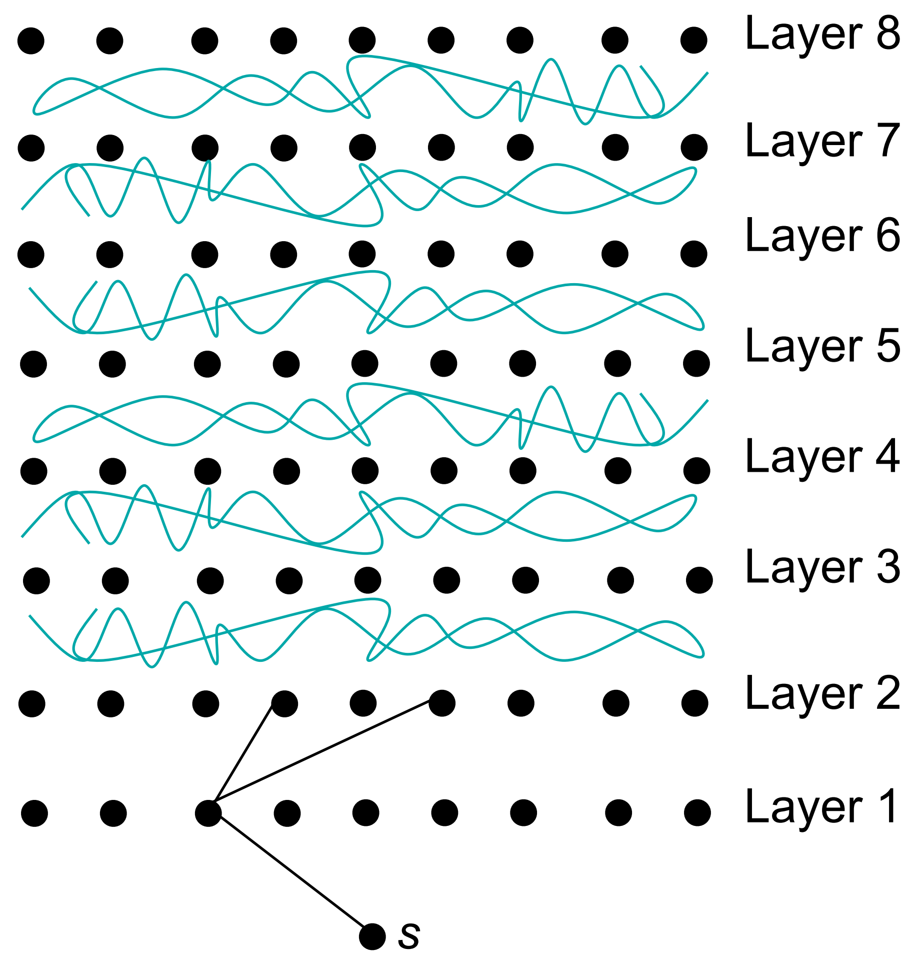

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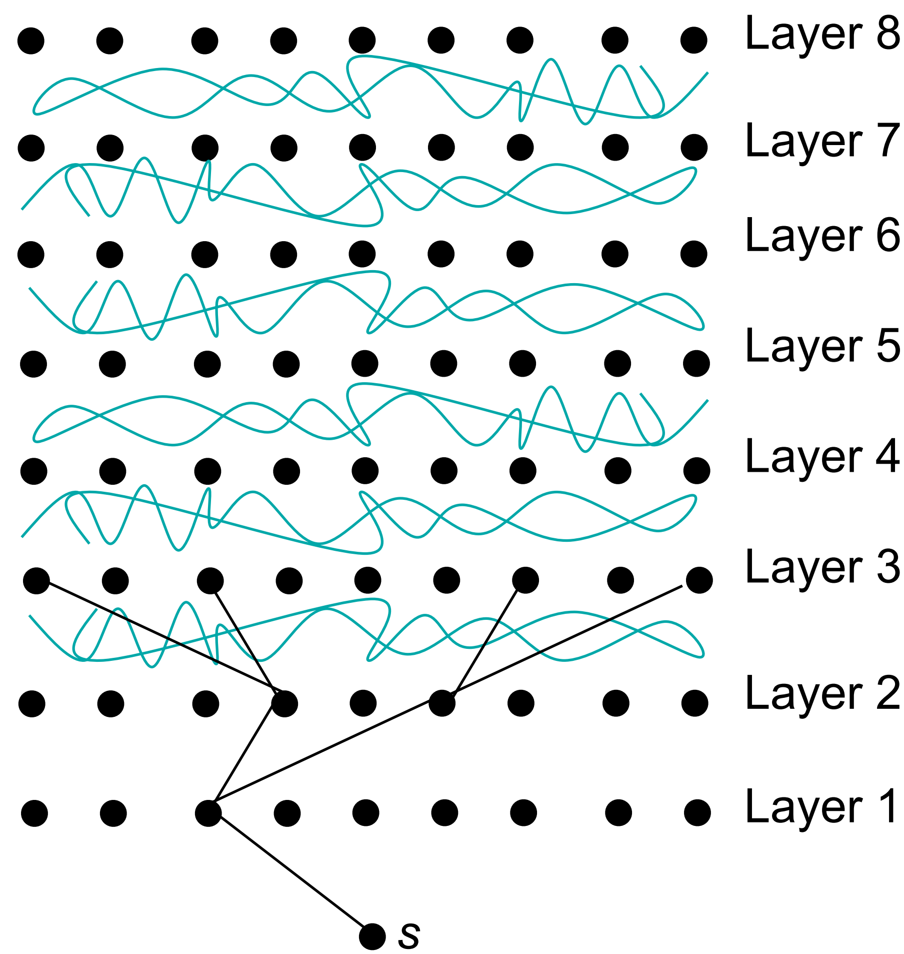

Specifics…

- Each vertex in layer *i* is connected to $({}^{n-1}/_{d})^{1/2d}$ vertices in layer *i+1* where *d* is the number of layers.
- Thm: For all $d \in \{1, \ldots, t/2\}$, doing a BFS takes *d* passes when the space restriction is $o(n^{1+1/t})$

Further Work?

- Estimating Distances in Multiple Passes?
- How important is the order of the edges?

Questions?

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