More on the Reliability Function of the BSC

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- No matter what code we use there is the possibility of making errors - for a given rate of transmission there is some degree of error that is inherent to the channel itself.

Making Decoding Errors

- Maximum Likelihood Decoding: When we receive a word y we'll guess that the sent codeword is the codeword that lies closest to it.
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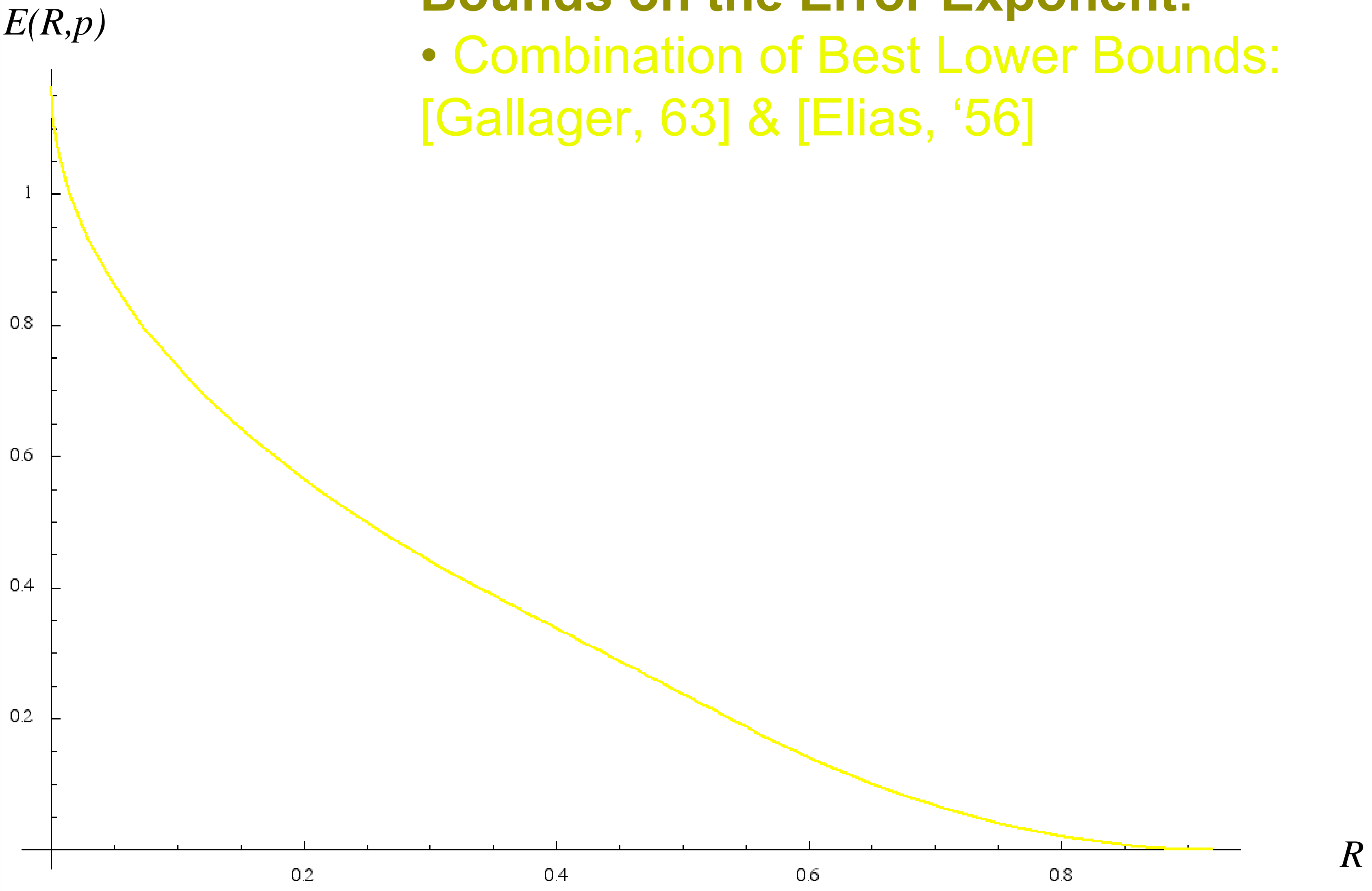
$$E(R,p) = -\lim_{n \to \infty} \frac{1}{n} \log\left[\min_{C: R(C) > R} P_e(C)\right]$$

Bounds on the Error Exponent:

- Combination of Best Lower Bounds: [Gallager, 63] & [Elias, '56]



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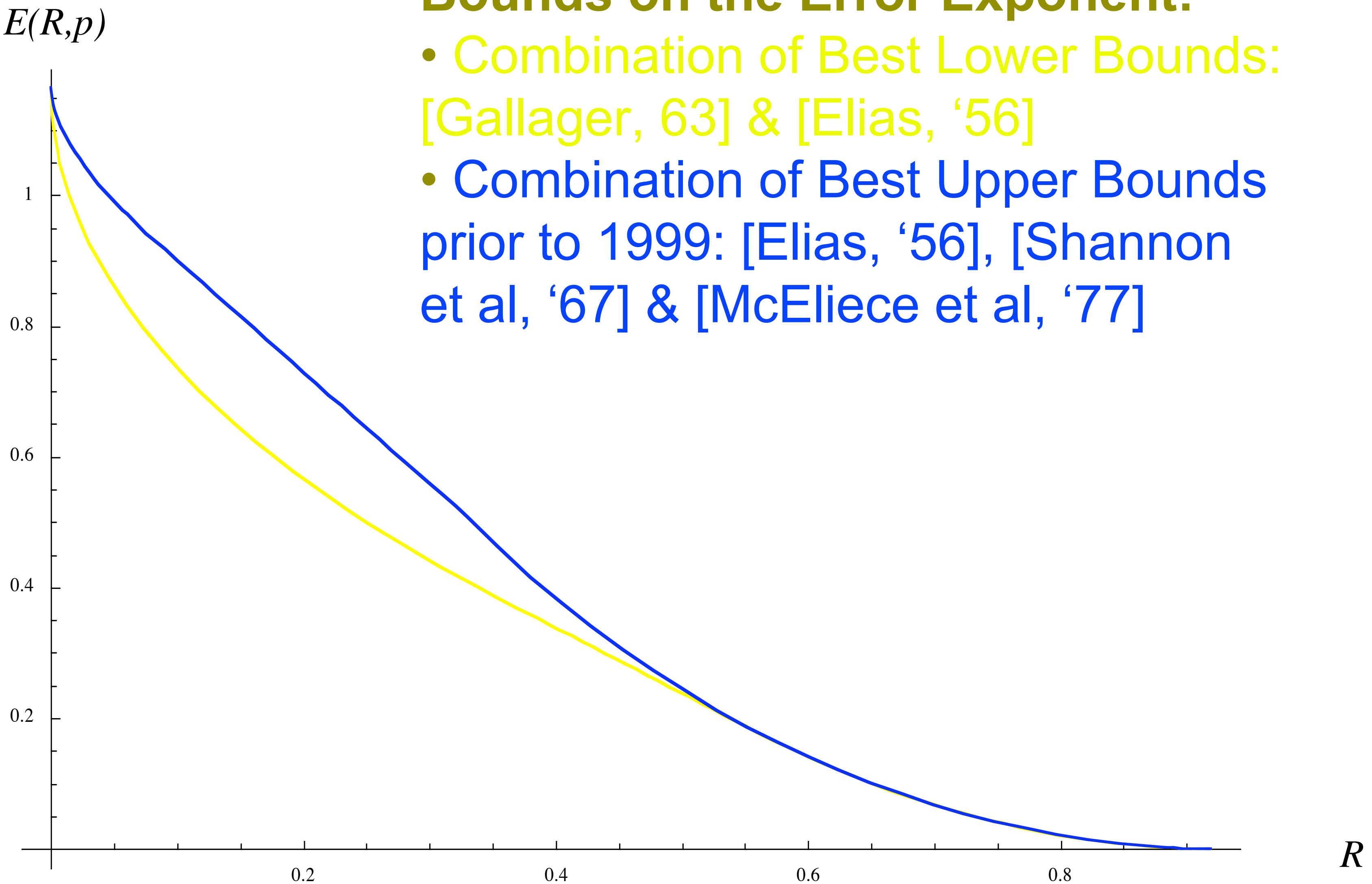

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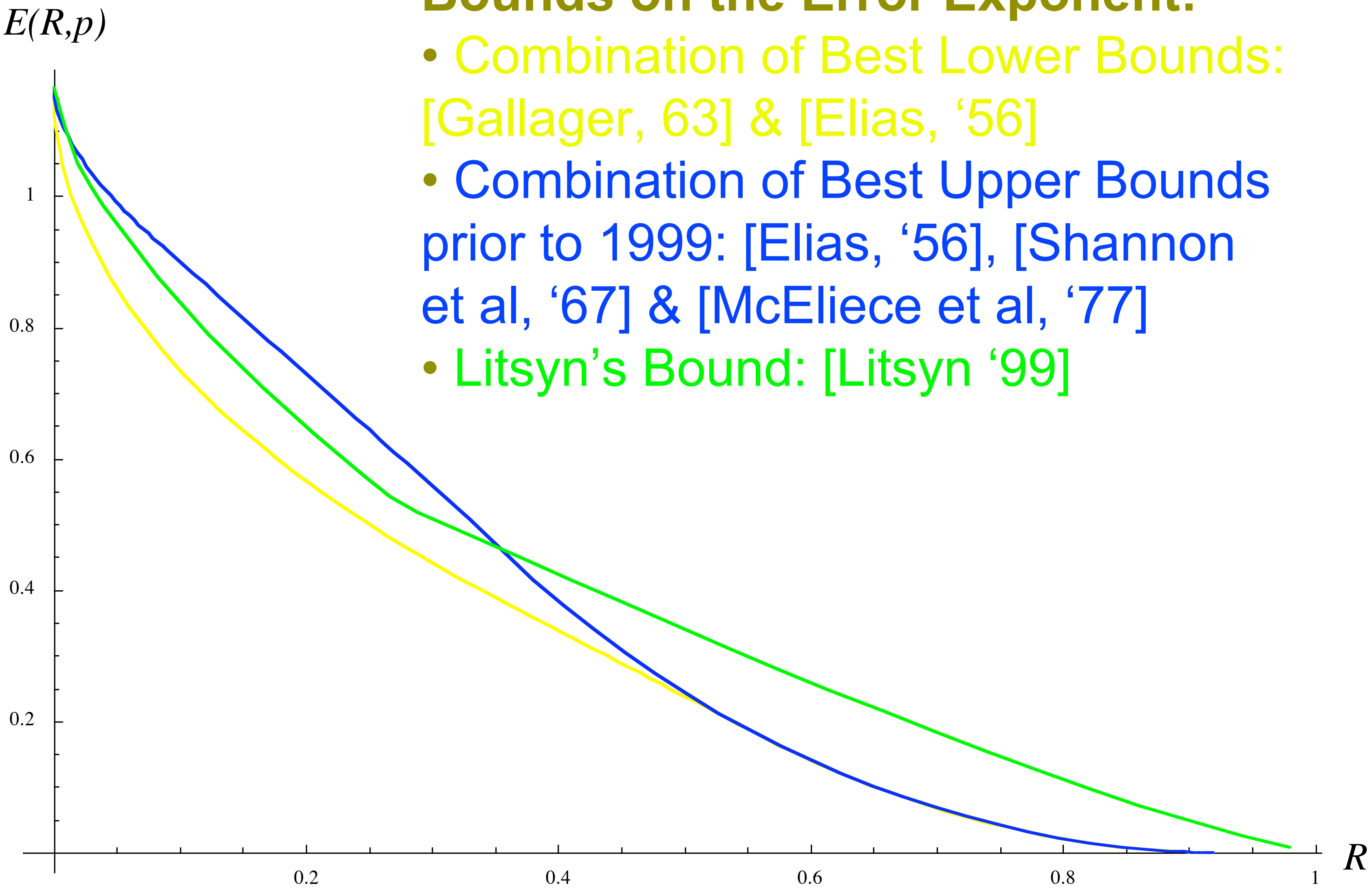

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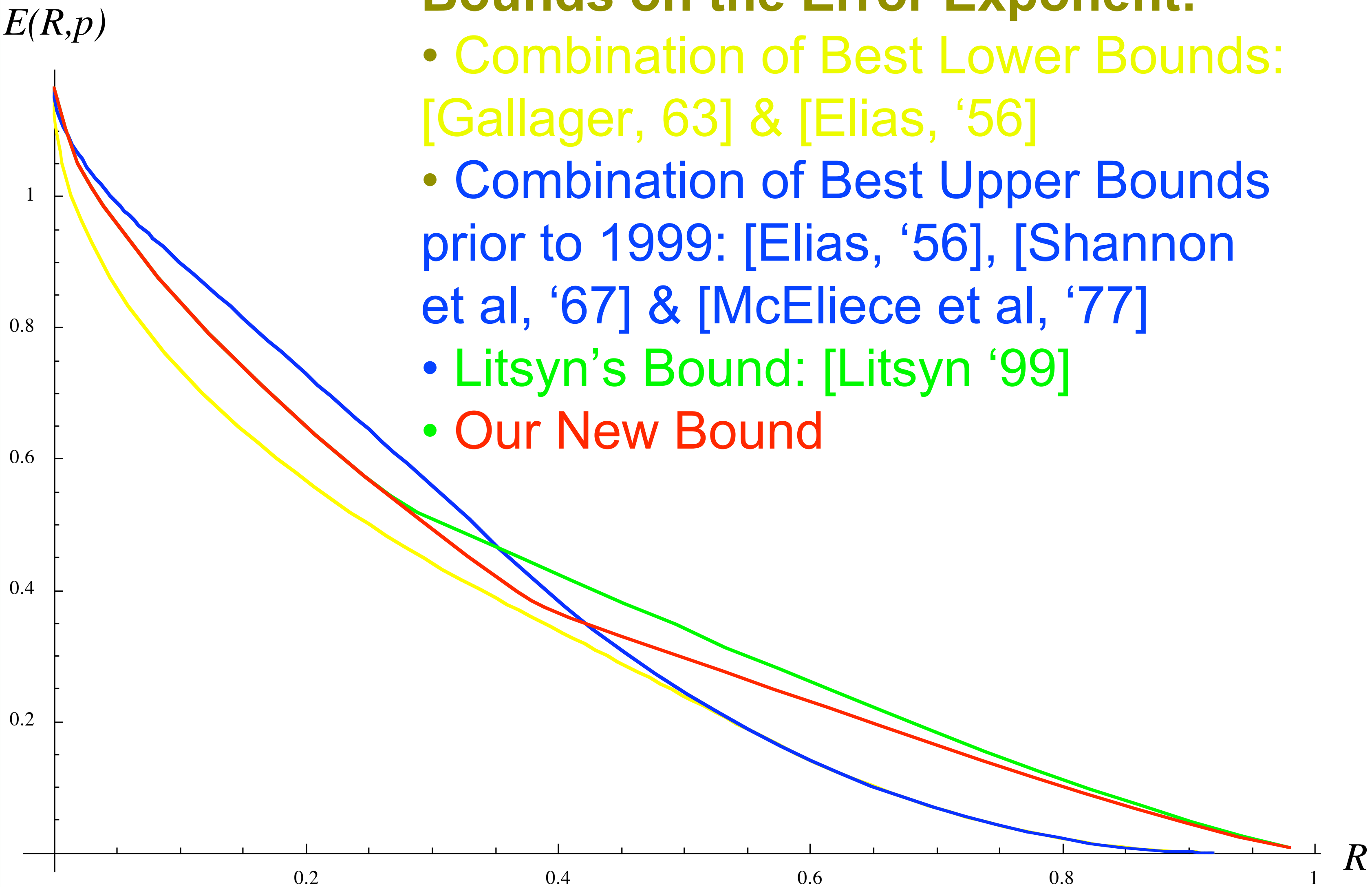

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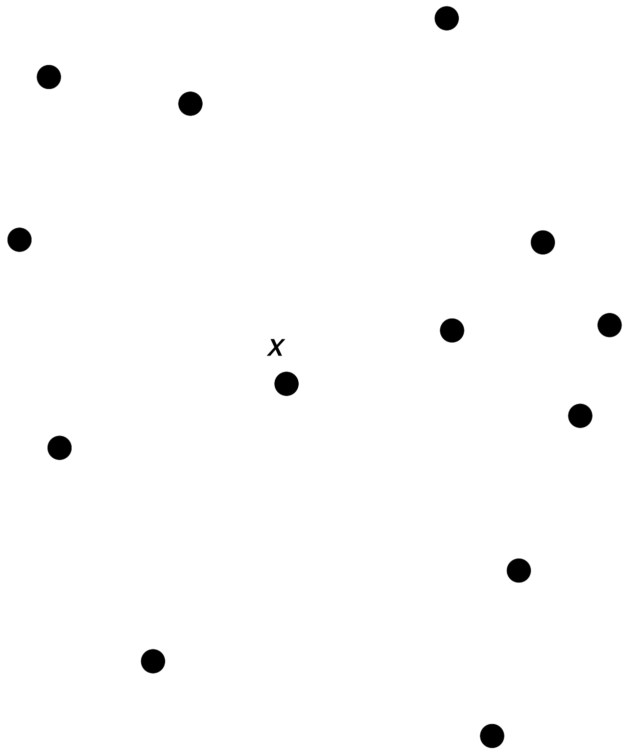
$$B_w(x) \geq \mu(R,w)$$

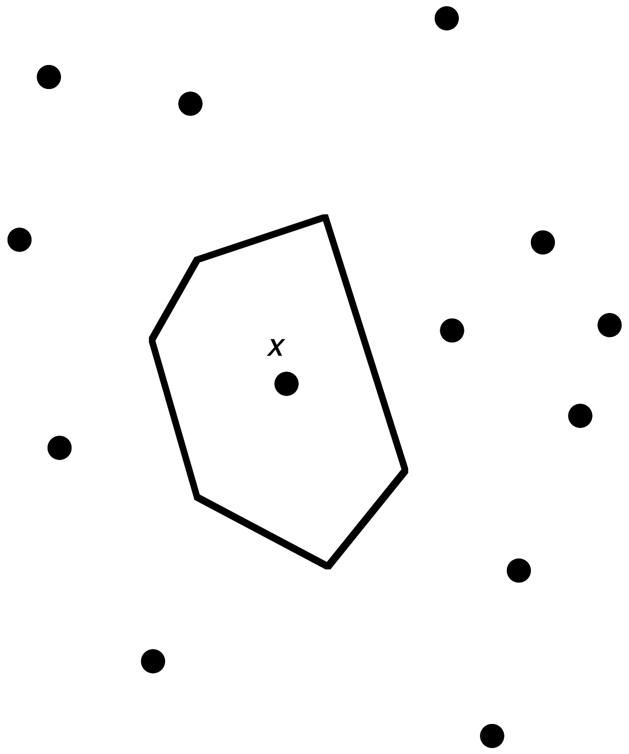
Estimating $P_e(x)$



$$P_e(x) = P_x(\{0,1\}^n \setminus D(x))$$

Estimating $P_e(x)$

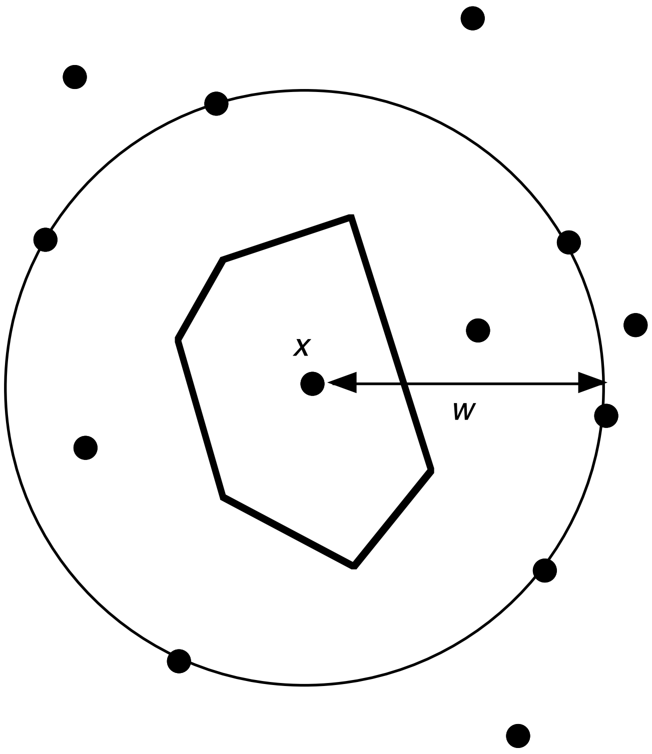
The Voronoi Region



$$P_e(x) = \sum_{y \in C:\, d(y,x_j) \le d(y,x) \text{ for some } x_j \in C} p^{d(y,x)} (1-p)^{n-d(y,x)}$$

Estimating $P_e(x)$

Use the distance distribution result…



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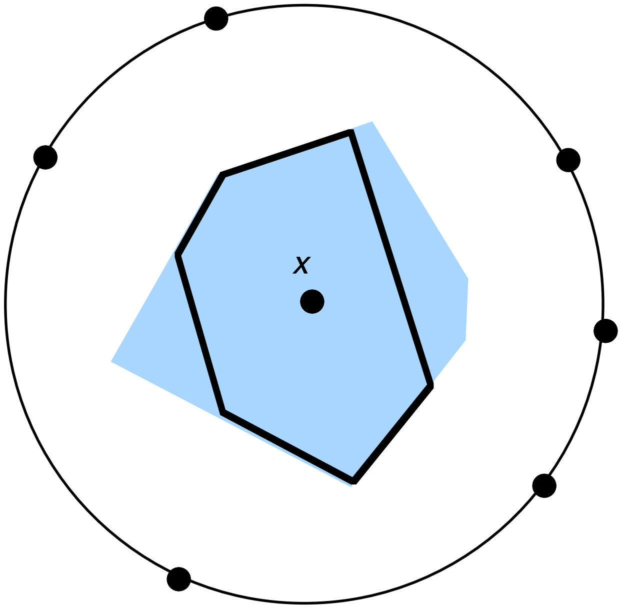
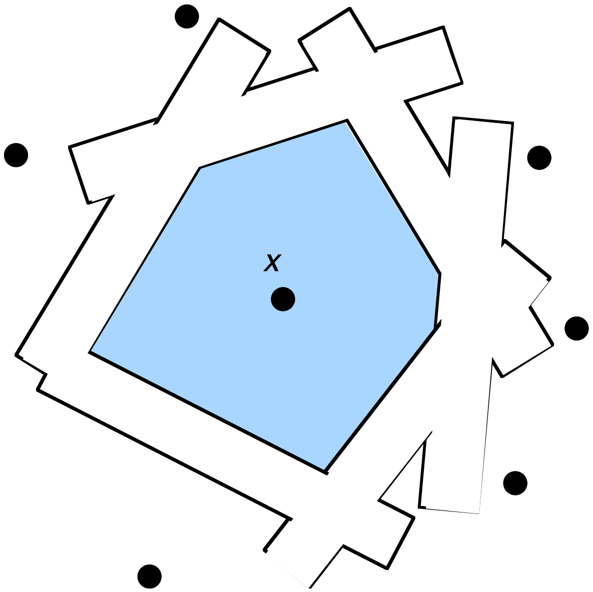
Approximating the Voronoi Region…



$$P_e(x) \geq \sum_{y\in C:\, d(y,x_j)\leq d(y,x) \text{ for some } x_j\in C \text{ where } d(x,x_j)=w} p^{d(y,x)}(1-p)^{n-d(y,x)}$$

Estimating *$P_e(x)$*

Introducing the X_j…



For each neighbour x_j define a set X_j such that

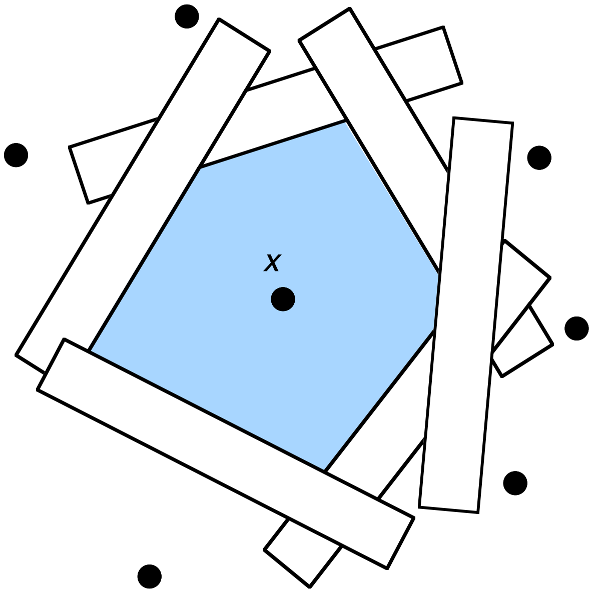
$$y \in X_j \Rightarrow$$

$$d(y,x_j) \le d(y,x)$$

$$P_e(x) \ge P_x\Big(\bigcup_{j:d(x,x_j)=w} X_j\Big)$$

Estimating $P_e(x)$

"Pruning" the X_j...



For each neighbour x_j assign a priority n_j at random. Let

$$Y_j = X_j \setminus \bigcup_{k:n_k>n_j} X_k$$

$$P_e(x) \geq \sum_{j:d(x,x_j)=w} P_x(Y_j)$$

Estimating $P_e(x)$

Applying the Reverse Union Bound…

The Reverse Union Bound:

Giving us our final shape of our bound:

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$$P_x(Y_j) = P_x(X_j \setminus \bigcup_{k:n_k>n_j} X_k)$$

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- Now look across the entire code. Let X_{ij} and Y_{ij} be the sets for the neighbourhood of codeword x_i.
- Therefore we have:

and

where, the amount of “pruning” is

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$$K_{ij} = \sum_{k:n_{ik}>n_{ij}} P_i(X_{ik} \mid X_{ij})$$

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- Either this is a “substantially” sized subcode or it isn’t.
- Ie, either we had to do a lot of pruning or we didn’t have to do a lot of pruning.

If S was not substantially sized…

- Just remove codewords in S from the code!
- Then in the remaining code we have for all Y_{ij}

$$P_i(Y_{ij}) \geq P_i(X_{ij})/2$$

- Hence, modulo constant factors, the average error probability satisfies

$$P_e(C,p) \geq A(w)\mu(w)$$

- where $A(w) = P_i(X_{ij})$

If *S* was substantially sized…

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where

- Consider a codeword x_j such that $K_{ij}>1/2$. Then there exists an l' such that

$$B_{l'}(x_j) > 1/(2nB(w,l'))$$

- The upshot of S being substantial is that we discover a nuisance level l_1, such that

$$P_e(x_j) \geq A(w)/B(w,l_1)$$

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Our Bound

- Continuing in this way we eventually get

$$P_e(C,p) \geq \min\left[A(w)\mu(w), \tfrac{A(l)}{B(w,l)}\right]$$

$$\text{where } 0 \leq l \leq w \leq \delta_{LP} n$$

- Minimizing over l and w gives us our final bound.

Random Linear Codes

- It can be shown that, with high probability, the weight distribution of a random linear code converges to

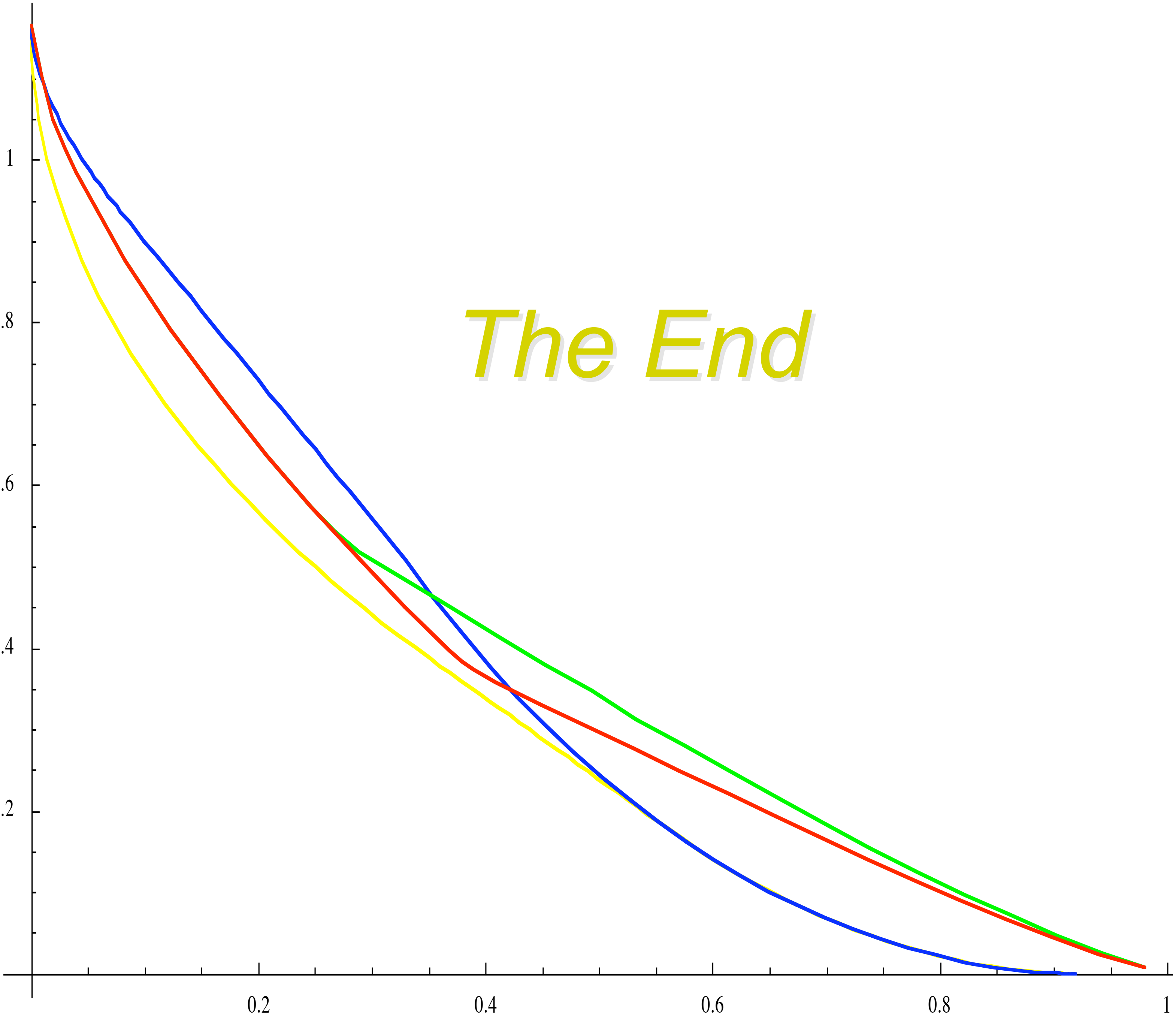
$$B_w = \exp[n(R+h(w)-1)]$$

- Using this instead of Litsyn's expression μ leads us to believe that the expurgation bound

$$E(R,p) \geq -\delta_{GV}(p)/2 \log 2p(1-p)$$

is tight for a random linear code for very low rates.

The End


1
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1